PRESERVICE PRIMARY TEACHERS'S NOTICING OF STUDENTS' GENERALIZATION PROCESS

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The objective of this research is to characterize levels of development of teaching competence in noticing students' mathematical thinking in the specific area of the generalization process. The findings provide descriptors of development levels of this teaching competence characterized by the way prospective primary teachers identified the relevant items in generalization processes from the answers given by primary students to generalization problems. The findings provide information for designing interventions in teacher training aimed at developing teaching competence in noticing students' mathematical thinking.

INTRODUCTION

Research on the professional development of mathematics teachers has highlighted the importance of teaching competence in noticing mathematical teaching-learning (Fernández, Llinares, & Valls, 2012; Mason, 2002; Jacobs, Lamb, & Philipp, 2010). Developing this teaching competence is one of the goals of teacher training programs and a relevant subject of study in research on mathematical education in recent years (van Es, & Sherin, 2002; Fernández, Llinares, & Valls, 2011).

Jacobs, Lamb and Philipp (2010) characterize this teaching competence using three skills mathematics teachers must develop: (a) identifying strategies used by the students; (b) interpreting student understanding; and (c) deciding what actions to undertake in the classroom. The research reported on here aims at providing information on how prospective primary teachers identify and interpret the mathematical thinking of primary school students in generalization processes. Generalization processes in this context are understood as linked to tasks in which the first terms of a succession are given in graphic form and students are asked: (a) to continue the succession; (b) to provide the number of elements making up the figures for distant terms; (c) to identify the rule; (d) to identify the position of a figure given the number of elements.

Research into how tasks of this sort are resolved by primary school students has shed light on the important role of the following components in the development of generalization processes:

Coordination between spatial and numerical structure: To extend a figural sequence, the students need to grasp a regularity that involves the linkage of two different structures: one spatial and the other numerical. From the spatial structure there emerges a sense of the elements' spatial position, whereas their numerosity emerges from a numerical structure (Radford, 2011; Rivera, 2010).

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Functional relationships: In order to identify a distant (or non-specified) term it is necessary to establish the relationship between the position of a figure and the number of elements that make it up.

Inverse process: To identify the position of a known figure it is necessary to establish a functional relationship that is the inverse of the above. Although many students are capable of establishing the relationship between the position of a figure and the number of elements making it up, they find it difficult to reverse the thinking (identifying the position of a figure when given the number of elements in it) (Warren, 2005).

In cases in which the functional relationship is a affine function, $f(n)=an+b,\ b\neq 0$, we must consider the *independent term* that appears as a constant in the function's expression. These elements play an important role in the development of the generalization process (Figure A).

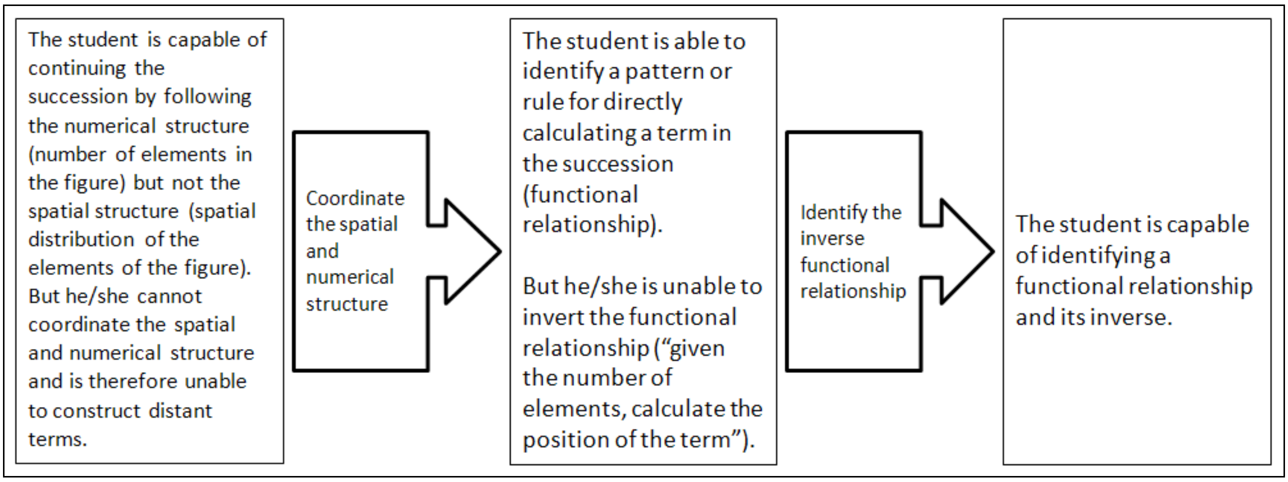


Figure A: Levels of development of the generalization process

The objective of this study is to characterize the development levels of the teaching competence of prospective primary teachers in noticing primary students' mathematical thinking in generalization process.

METHOD

Participants

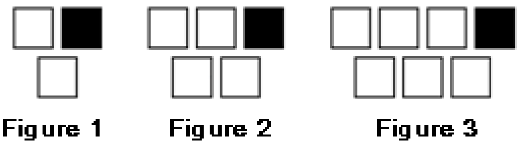
The participants were 40 prospective primary teachers (PPTs) in the second semester of their academic program studying subjects focused on primary students' development of a numerical sense.

Instrument

Based on prior research on the development of the generalization process in primary students (Radford, 2010; Carraher, Martínez, & Schliemann, 2007) we devised a questionnaire made up of the responses of three students to three problems displaying a succession of figures that follow a pattern of additive growth (Figure B).

Problem 1

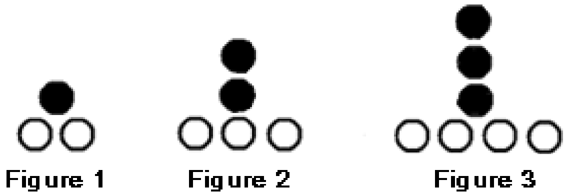
Observe the following figures:



1. Continue the succession and draw figures 4 and 5.
2. Without drawing figure 25, can you tell how many squares it would have? Explain how you figured this out.
3. How would you calculate the total number of squares for a given figure?

Problem 2

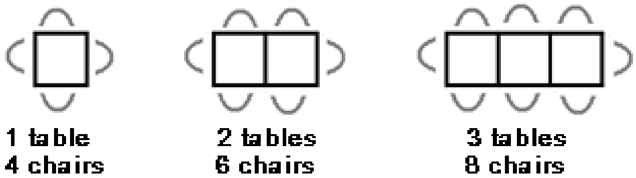
Observe the following figures:



1. Continue the succession and draw figures 4 and 5.
2. Without drawing figure 30, can you tell how many circles it would have? Explain how you figured this out.
3. How would you calculate the total number of circles for a given figure?

Problem 3

Observe the following figures representing tables and chairs:



As you can see, we have put 4 chairs around one table, 6 chairs around two tables, and 8 chairs around three tables.

1. Can you draw 4 tables and the number of chairs it should have?
2. How many chairs can we put around 5 tables in this way? And around 6 tables?
3. For a party we put 18 tables together along with the appropriate number of chairs. How many guests will be able to sit? Explain how you found your answer.
4. If there are 42 children invited to a birthday party, how many tables will we need to put together in a row? Explain how you found your answer.
5. Explain in your own words a rule connecting the number of tables and the number of chairs.

Figure B: Problems solved by primary students

In problems 1 and 2 the rule is $2n+1$, n being the number of the figure. In the first problem the independent term corresponds to the only black square in each figure, and in the second to the difference between the number of white and black circles. The rule for the third problem is $2n+2$.

The answers of the three students to the three problems were chosen based on different levels of development of the generalization process (Figure A) (Radford, 2011; Warren, 2005):

Student A's answers to the three problems show a generalization process development that allows him to continue the succession for close terms, obeying the quantitative growth pattern but not the figures' spatial structures. However, the student is unable to construct the distant terms, as he/she cannot coordinate the spatial and numerical structure of the figures and ignores the independent term.

Student B's answers to the three problems point to a generalization process development that allows him/her to coordinate the spatial and numerical patterns, recognize functional relationships in specific cases, and express the rule with their own words as a functional relationship. However, the student is not able to invert the functional relationship in specific cases (without the inverse process).

Student C's answers add to the above skills the ability to invert the functional relationship in specific cases (with the inverse process).

The PPTs were asked to respond to the following three questions:

1. *What aspects of student E's answers with respect to each of the problems would you stress, indicating to what problem you are referring.*
2. *Based on the aspects you have pointed out, identify characteristics of the generalization process of student E for the three problems.*
3. *Given the characteristics of the generalization process you listed in the above point, if you were a teacher, what would you do to improve this process?*

In this article we discuss the findings of the analysis of the first two questions.

Analysis

The analysis was carried out in two phases. The first analyzed the responses of each PPT to the first two questions from the questionnaire. The objective of analyzing the first question was to see to what extent the PPTs identified the significant mathematical elements used by the students in solving generalization problems. In analyzing the second question, we examined to what extent the PPTs identified characteristics of the development of the generalization process of each of the primary students.

The goal of the second phase of the analysis was to generate descriptors of different degrees of development of teaching competence in noticing students' mathematical thinking in the realm of generalization process development. To do so, we jointly examined the mathematical elements and the characteristics of development of the generalization process that each PPT had identified. We used an inductive procedure for generating categories in which the results of the various steps were contrasted independently by three researchers, who discussed any initial discrepancies. Based on a preliminary analysis of a sample of answers we generated an initial system of categories to bring to light aspects that could be considered relevant to teaching competence in noticing students' mathematical thinking in the generalization process. As a result of this process we generated descriptors on three levels of the development of noticing:

Level 1. PPTs identify that the student continues the succession for close terms, obeying the pattern of quantitative growth but not the figures' spatial structure.

Level 2. PPTs identify that the student coordinates the spatial and numerical scheme, that he/she recognizes the functional relationship in specific cases, and that he/she is able to express the rule as a functional relationship.

Level 3. PPTs identify that the student coordinates the spatial and numerical scheme, that he/she recognizes the functional relationship in specific cases and is able to express the rule as a functional relationship, and that he/she can invert the functional relationship in specific cases.

FINDINGS

Our analysis has enabled us to classify 35 of the 40 PPTs (87.7%). The responses to the questionnaire by 5 of the PPTs could not be classified into any of these levels as they did not identify the generalization level of any student (2 PPTs) or they did not identify the most elementary level (3 PPTs identified only the two students who expressed the rule).

	Level 1	Level 2	Level 3
PPTs	10	12	13

Table 1: Number of PPTs on each of the levels identified

Level 1. PPTs identify that the student continues the succession for close terms, obeying the pattern of quantitative growth but not the figures' spatial structure (10 PPTs).

The PPTs on this level identified only the case of the student that continues the succession for close terms and has difficulty in calculating the number of elements of a distant term because he/she cannot coordinate the spatial and numerical pattern of the succession and omits the independent term. For example, one PPT, when referring to solving problem 1, mentioned the inability to coordinate spatial and numerical structures and that the student ignored the independent term:

> The exercises are wrong, since he/she did not follow either the numerical or spatial order and the result is not correct. The strategy the student used of multiplying the two rows would not be wrong if he/she added one to the multiplication (…). The difficulty the student had was because he/she didn't realize that you have to add the black square.

The PPTs at this level of development of teaching competence did not recognize the characteristics of the generalization process in the responses of the other students. Thus, they did not mention the functional relationship when interpreting the answers of the two students who successfully expressed the rule (B and C).

Level 2. PPTs identify that the student is able to coordinate the spatial and numerical structure, that he/she recognizes the functional relationship in specific cases, and that he/she can express the rule as a functional relationship (12 PPTs).

The PPTs at this level were capable of identifying and differentiating the students who were able to express the rule. Nevertheless, they did not find it relevant whether the student was able to carry out the inverse process. Therefore, they did not recognize the difference in the generalization process involving the ability to invert the functional relationship in specific cases. For instance, one PPT, when referring to the answers of the student who was unable to carry out the inverse process (B) omitted this aspect:

> Student B adequately resolves each step. One could say that he/she follows and maintains the spatial and numerical structure. He/she describes verbally what guidelines or pattern should be followed.

Level 3. PPTs identify that the student coordinates the spatial and numerical pattern, that he/she recognizes the functional relationship in specific cases and expresses the rule as a functional relationship, and is able to invert the functional relationship in specific cases (13 PPTs).

The PPTs at this level were able to identify the different characteristics of the generalization process in the answers of the three primary students.

One PPT, for example, interpreted the solution of problem 1 by the student who was limited to continuing the succession for close terms (A), mentioning the lack of coordination between the spatial and numerical structures and considering that this was the reason that he/she ignored the independent term:

> The answer to this problem was incorrect. In carrying out this activity, the student did not take into account the figures' spatial distribution, but did include the numerical distribution (…) It is possible that when doing the second step he/she did not see the black square and therefore did not count it.

The PPT described the characteristics of student B's generalization process indicating that he/she was able to express the rule but cannot carry out the inverse process:

> In the first and second exercises he/she is able to make a generalization globally, since the student has no problem in resolving all the steps. In the third exercise he/she also successfully finds the general pattern of the problem but does not manage to apply it the other way around.

And the PPT identified the realization of the inverse process as a differentiating trait of comprehension of the generalization process between students B and C, since both coordinate spatial and numerical structures and identify the global pattern, but only student C is capable of applying the inverse process in specific cases:

> He/she correctly carries out the generalization of step (1), following the pattern of the question's figures and obeying their spatial and numerical distribution. The student is also able to make a more far generalization since by using a strategy of spatial counting he is able to solve problems with higher numbers without difficulty. Finally, this student is capable of making an global generalization, since he/she is able to see the pattern followed by all the figures.

DISCUSSION

The aim of this research is to characterize levels of development of teaching competence in noticing students' mathematical thinking in the generalization process. The results have given us descriptors of the development of this teaching competence characterized according to how prospective primary teachers identified the elements relevant to the generalization processes from the answers given by primary students to generalization problems.

One finding of this study is that PPTs have trouble interpreting the mathematical thinking of students in the development of the generalization process. The fact that some PPTs identified only the characteristics of the generalization in specific cases and were unable to recognize other characteristics of the generalization process displayed in the primary students' answers could be due to a deficiency in their mathematical knowledge. This fact showed through because the PPTs at level 1 of noticing development produce a generic discourse that does not reflect the relevant components that are integral to the generalization process. By contrast, the discourse generated by the PPTs who were capable of recognizing when the primary students had expressed a rule was more specific.

Another finding from this study is the characterization of three levels of development of this teaching competence. Moving up from level 1 to 2 occurs when the PPTs are capable of identifying as relevant the functional relationship when interpreting the responses of primary students. The step from level 2 to 3 takes place when the PPTs are able to recognize as relevant to the generalization process the realization of the inverse process based on identifying the functional relationship.

These findings provide information for designing materials for teacher training that take into account the characteristics of the PPTs learning. In this respect, the instrument designed can be a springboard for devising teaching materials in teacher training programs whose objective is the development of skills in identifying the mathematical components that are relevant in solving these types of tasks and interpreting the students' answers.

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