TOWARDS A TEACHING APPROACH FOR IMPROVING MATHEMATICS INDUCTIVE REASONING PROBLEM SOLVING

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The study aimed at proposing and assessing a training program that integrates both inductive reasoning problem solving and the development of mathematical concepts. This approach was developed on the basis of a general theory of inductive reasoning, which delineates six related classes of problems and the corresponding solution processes and it was implemented to sixth grade students. Data were collected through a written test consisted of mathematics problems of the six structures. Three repeated measurements were conducted with a break of 3-4 months between them. Findings revealed a significant improvement of mathematics inductive reasoning problem solving of the trained students while the training effect persisted for at least four months after the implementation of the program.

INTRODUCTION AND THEORETICAL BACKGROUND

Inductive reasoning is the highest-order cognitive skill that characterizes learning potential. It is considered to be a central component of critical thinking and one of the basic learning skills that contributes to problem solving (Haverty, Koedinger, Klahr, & Alibali, 2000). It is defined as the process of inferring a general rule by observation and analysis of specific instances (Haverty et al., 2000), and therefore it is a vital process for everyday life and for scientific investigation in particular.

In mathematics education inductive reasoning is enclosed among the most important goals of the curriculum (NCTM, 2000). It is closely related to the exploration and the generalization of different kinds of patterns that serve the basis of structural knowledge in mathematics learning (Johnassen, Beissner, & Yacci, 1993). As a generalization process, it is also fundamental to the development of many mathematical concepts, especially in algebraic concepts and in problem solving situations (Haverty et al., 2000; Orton & Orton, 1994; Warren, 2006). Consequently, mathematics teaching should focus on fostering basic skills in generalizing, and expressing and systematically justifying generalizations (Kaput & Blanton, 2001), as well as in developing strategies for solving various types of inductive reasoning mathematics problems. Nevertheless, in elementary education there is little emphasis on inductive reasoning as object of study; rather it is considered that could be developed as a by-product of the teaching of the content as defined in traditional curriculum (Hamers, De Koning, & Sijtsma, 1998). Classroom activities usually focus on mathematical products rather on mathematical processes and strategies. Inductive reasoning problems are likely to be marginalized by the press towards

2009. In Tzekaki, M., Kaldrimidou, M. & Sakonidis, H. (Eds.). *Proceedings of the 33rd Conference of the International Group for the Psychology of Mathematics Education*, Vol. 4, pp. 313-320. Thessaloniki, Greece: PME.

computational skills (Thompson, Philipp, Thompson, & Boyd, 1994) or appear in an abbreviated, arithmetic form (Blanton & Kaput, 2005). Consequently, many students have a lot of difficulties in solving problems.

Considering the valuable aspect of inductive reasoning in learning, research studies focused on designing teaching programs for improving inductive thinking in schooling (Klauer & Phye, 1994). Although these programs were oriented towards thinking processes and promoted inductive reasoning as a tool for problem solving, they were developed in a general content domain using content-free and daily life problem formats. Even in mathematics education, where inductive reasoning is an important process to investigate the gaining of a deeper understanding of mathematical cognition (Haverty, et al., 2000), there is a lack of the appropriate guidelines for designing comprehensive content-related approaches for improving inductive thinking within the content of the mathematics curriculum.

The present study, attempted to apply a teaching approach that integrate inductive reasoning problem solving procedures in a mathematics concept-development context. The proposed approach is based on a prescriptive theory of inductive reasoning (Klauer & Phye, 1994) while it is in line with the pedagogical principles and methods presented in the literature. The focus of the study was twofold: (a) to investigate whether the proposed approach improved students' mathematics conceptual knowledge and their ability to solve mathematics inductive reasoning problems of different structures, and (b) to specify the nature of change of students' mathematics inductive reasoning through the passing of time.

A prescriptive theory of inductive reasoning

Klauer (Klauer & Phye, 1994) suggests that inductive thinking could be improved through the teaching of the steps of an induction process that are necessary and sufficient to arrive at a generalization. These essential steps are resulted from an analytic definition of inductive reasoning, which delimits inductive reasoning problems from other types of problems (e.g. deductive) specifying their cognitive solution processes. This definition considers that inductive reasoning is the systematic and analytic comparison of objects aiming at detecting similarities and/or differences among them with respect to attributes or relations. It also presupposes that all types of inductive reasoning problems could be classified into two main subsets, the group-problems and the row-problems. The group-problems are dealing with attributes while the row-problems are dealing with scanning relations. Each subset comprises three different types of problems that could be discriminated in terms of the cognitive processes needed for their solution. That is, the group-problems set includes: (a) problems that require finding similarity of attributes among objects, (b) problems that are related with noting differences among objects with respect to attributes; and (c) problems that require a determination of both common and different attributes of objects. The row-problems set involves: (a) problems that require finding similarity among relationships; (b) problems that require detecting differences in relationships; and (c) problems that require finding either equivalence or dissimilarity of relationships. From a teaching perspective, the essential steps that are necessary for a successful inductive reasoning problem solving are the followings: First to train students to recognize the reasoning structure of an inductive task (group or row structure scheme), and then to apply the appropriate cognitive solution process. According to Klauer the mastery of these steps will improve students' ability to solve any type and complexity of inductive reasoning problems.

The proposed approach

The proposed teaching program was designed on the basis of a mathematics cognitive framework of inductive reasoning that delineates students' abilities in solving various mathematics problems that all require the use of inductive reasoning (Christou & Papageorgiou, 2007). These abilities correspond to the cognitive processes of similarity, dissimilarity, and integration, and are associated with the level of attributes and the level of relations, which specify the aspects that are compared in a mathematics inductive task. Thus, instruction aimed at helping students to distinguish whether a problem involves relations or attributes (reasoning structure) and then to identify whether there is a need to find similarity or difference or both similarity and difference (integration) in the attributes or in the relations involved in the problem (processing structure), in order for the problem to be solved.

In line with Klauer's training program, instruction proceeded through three hierarchical levels that correspond to three successive phases of knowledge development: (a) the conceptual-analytical level that corresponds to the development of the declarative knowledge; (b) the procedural level that corresponds to the development of the procedural knowledge; and (c) the strategic level, which is related to the development of the metacognitive knowledge. These three levels of instruction overlapped and proceeded developmentally throughout the time the program was carried out.

The conceptual-analytical phase aimed at the conceptual recognition of the different structures of inductive reasoning problems. Thus, instructional activities asked students to classify given problems into two main groups in terms of their reasoning structure, i.e. the kind of the objects needed to be compared (attributes or relations), and then to identify the different problem-formats included in each group. Teaching at this phase emphasized cooperation and discussion between students in order to facilitate discrimination of the various types of problems and to localize similarities and/or dissimilarities between different problem-formats. Furthermore, students encouraged to construct concept maps and diagrams to represent the relations existed among different problem-formats with respect to their solution comparison processes (processing structure) and then to relate unknown problems to the previous ones by analogy.

The procedural phase aimed at teaching students how to solve inductive reasoning problems of different structures. Problem solving at this phase involved also the attainment of new knowledge. Thus, instructional activities focused on constructing cognitive schemes of the new concepts that could be infused in an inductive processing structure. For facilitated learning, worked-examples were mainly used to relate problems involved new concepts to familiar ones by analogy. Furthermore, worked-examples and flowcharts were used to demonstrate the solution steps of each problem structure an to model problem solving processes in order for the students to localize similarities and/or differences between them.

Finally, the strategic phase aimed at accelerating the spontaneous application of the six reasoning processes in solving inductive reasoning mathematics problems. Therefore, the training activities encouraged students to solve problems of different representations and complexity, to describe the solution processes and to justify their thinking procedures in terms of the structures of the problems.

The problems used during the training were derived from 6th grade mathematics curriculum and were related to attributes and relations of numbers and geometrical figures. Examples of some concepts intended to be developed through the proposed approach were: (a) the properties of numbers and numbers' operations, such as the multiplication of the odds and even numbers, (b) the attributes and features of the 2D and 3D geometrical figures, (c) numerical proportions, and (d) various types of number-sequences, like the sequence of the squared or the triangular numbers, arithmetic and geometric sequences, as well as patterns with geometrical figures.

METHOD

Participants and procedure

The sample of the study consisted of 137 sixth grade students (63 boys and 74 girls), from seven existing classes at elementary schools in an urban area of Cyprus. The study was based on an experimental-control design, thus sixty students comprised the experimental group while the rest comprised the control group. Students were assigned to the two groups according to their performance on mathematics inductive reasoning problem solving on the first measurement, which was carried out at the beginning of the study. The two groups were of the same performance.

Data were collected through a written test that measured students' performance on mathematics inductive reasoning problems of the six different structures delineated by the cognitive mathematics inductive reasoning framework. The same test was administered to students three times. The interval between successive administrations was 3-4 months.

Students of the experimental group were involved in the activities of the proposed program after the first measurement. The duration of the intervention was twelve 40-minutes teaching periods that were spread over nine weeks during the regular mathematics lessons. Students of the control group did not have any systematic instruction in inductive reasoning.

The assessment instrument

Twenty-one problems of six different structures comprised the test developed to measure mathematics inductive reasoning problem solving ability (see Table 1). Students had 60 minutes to complete the test. The scale reliability of the whole questionnaire was found to be very high. The Cronbach's alpha coefficients for the whole set of problems were 0.85, 0.88, and 0.88, at first, second and third measurement, respectively.

<table>
<tr><th></th><th>Attributes tasks</th><th>Relations tasks</th></tr>
<tr><td>Similarity</td><td>The numbers below have something in common. Write the common feature of the numbers:
4, 16, 8, 32, 20, 100, 40</td><td>Complete with the right number.
1 5 13 29 ……
(a) 33 (b) 37 (c) 45 (d) 61</td></tr>
<tr><td>Dissimilarity</td><td>Find the numbers that does not fit in with the others and put it in a circle. Explain.
9 21 11 15 12 6 23</td><td>One of the figures disturbs the sequence. Find it and define the right sequence.
</td></tr>
<tr><td>Integration</td><td>Write the number 24 in the appropriate cell. Explain.
<table><tr><td>6 18
12</td><td>16 8
2</td></tr><tr><td>3 15
9</td><td>7 25
5</td></tr></table></td><td>Complete the empty cell with the appropriate number.
<table><tr><td>8</td><td>4</td><td>2</td></tr><tr><td>1</td><td>$\frac{1}{2}$</td><td>$\frac{1}{4}$</td></tr><tr><td>$\frac{1}{8}$</td><td>$\frac{1}{16}$</td><td>;</td></tr></table></td></tr>
</table>

Table 1: Examples of problem formats used in the test

Data analysis

A univariate analysis of covariance (ANCOVA) was implemented to the data in two parts. Initially, ANCOVA was used to examine whether the proposed approach improved students' ability to solve the whole set of problems included in the test. Thus, post-test attainments were used as depended variables, while the corresponding pre-test scores were used as covariates. Then, this kind of analysis was carried out to investigate the durability of the training effect; therefore it is preceded on comparing the scores of the two groups revealed from the third measurement to their pre-test attainments.

Finally, a multivariate analysis of covariance (MANCOVA) was used to explore students' improvement on solving each of the six different problem-formats as well as problems of the same reasoning (attributes and relations problems) or processing structure (similarity, dissimilarity and integration problems). Thus, in this case we set as depended variables the post-test scores related to each of the six kinds of problems or to each subset of problems, and as covariates we regarded the corresponding pre-test scores.

RESULTS

Regarding the whole set of mathematics inductive reasoning problems, results showed that students who received training outperformed the control group at the post-test ($F_{(1, 134)}=32.779$, $p<0.05$). This indicates that the training program has a positive impact on students' mathematics inductive reasoning problem solving after intervention. Furthermore, the training effect could be persistent at least four months after training, as revealed from the outcomes of the second part of the ANCOVA ($F_{(1, 134)}=5.254$, $p<0.05$). Table 2 presents the statistic indices resulted from the two-phase analysis of covariance.

Measurement	Group of Students	$\overline{X}$	Sum of Squares	F	p-value
Second	Experimental	0.871	0.641	32.779	<0.05
	Control	0.731			
Third	Experimental	0.768	0.149	5.254	<0.05
	Control	0.701			

Table 2: Students' attainments on the second and third measurements

With respect to each of the six kinds of problems, the results of the MANCOVA showed that the experimental group of students performed significantly better than the control group on the four kinds of problems after the implementation of the intervention (Pillai's $F_{(6, 129)}=9.051$, $p<0.05$). Specifically, the significant differences between the two-group performances were observed on attributes-similarity problems and on attributes-dissimilarity problems as well as on relations-similarity problems and relations-dissimilarity problems (see Table 3). Despite the improvement of the experimental group on solving both attributes-integration and relations-integration problems, this improvement was not significantly higher than the control group's ability. From a broader perspective, findings also revealed that the experimental group performed significantly higher than the control group on all the five sets of problems that formed on the basis of their reasoning structure (attributes and relations problems) (Pillai's $F_{(2,133)}=14.506$, $p<0.05$) or their processing structure (similarity, dissimilarity and integration problems) (Pillai's $F_{(3,132)}=14.372$, $p<0.05$). Given that the mathematics inductive reasoning of the two groups were equal at the beginning of the study ($t_{1,135}=-1.948$, $p>0.05$), these results indicate that the training effect have a positive impact on students' ability to solve various types of problems. Furthermore, the training effect contributed to the deeper understanding of the mathematical concepts involved.

Problem's Structure	Group of Students	Mean	SD	Sum of Squares	F	p-value

Attributes Problems	Similarity	Experimental	0.920	0.209	1.118	30.334	<0.05
		Control	0.782	0.290			
	Dissimilarity	Experimental	0.817	0.379	1.075	8.603	<0.05
		Control	0.708	0.424			
	Integration	Experimental	0.758	0.298	0.194	2.186	>0.05
		Control	0.714	0.329			
Relations problems	Similarity	Experimental	0.908	0.145	0.149	6.140	<0.05
		Control	0.867	0.197			
	Dissimilarity	Experimental	0.792	0.284	1.986	34.279	<0.05
		Control	0.584	0.321			
	Integration	Experimental	0.883	0.192	0.135	3.230	>0.05
		Control	0.847	0.247			

Table 3: Students' post-test attainments on the six types of problems

CONCLUSIONS

In this study we attempted to design a teaching model for developing inductive thinking within the content of the school mathematics in real-classroom situations. Findings revealed that this approach is effective since it improves students' ability to solve mathematics inductive reasoning problems of various structures. The use of the worked-examples in developing both the conceptual and the procedural knowledge of inductive reasoning mathematics problem solving seemed to guide students in their schemes construction and therefore in learning. This supports the idea that inductive reasoning problem solving and concept development could be integrated through appropriate training in a regular mathematics lesson. Though improvement was observed on all the six problem-formats included in instruction, the size of the training effect was different according to the problems' complexity. Integration problems seemed to need much more practice than the other types of problems for gaining spontaneous application of the solution procedures; they require the simultaneous application of two cognitive strategies and therefore they demand much more capacity of working memory to proceduralize the combination of the associated cognitive schemes, especially when they involve newly acquired concepts.

Nevertheless, the study is of great importance because incorporates problem solving and concept-development in real-classroom instruction. Thus, this approach could be used as a tool in teachers' instruction for developing inductive reasoning mathematics problem solving as well as the developing of specific mathematical concepts. Also, it could be used as a prototype for designing instructional programmes for improving thinking skills within the different subjects of the school curriculum.

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