ANALOGICAL REASONING BY THE GIFTED

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The powerful role of analogical reasoning in discovering mathematics is well substantiated in the history of mathematics. Mathematically gifted students, thus, are encouraged to learn via in-depth exploration on their own based on analogical reasoning. In this study, 4 gifted students (two in the 6th grade and two in the 8th grade) were asked to formulate or solve mathematical problems through analogical reasoning. All 4 students produced fruitful constructs led by analogical reasoning; however, findings showed there to be different tendencies in students' analogy and mathematical construction. In addition, findings suggested different educational needs and support are necessary for the gifted.

INTRODUCTION

Polya (1954, 1962) proclaimed analogy an essential mathematical reasoning ability, and studies on the meaning and development of analogical reasoning (English, 2004; Alexander et al., 1997) have shown how important analogy is in the development of logical thinking. The powerful role of analogical reasoning in the construction of new mathematics is also well documented in the history of mathematics. Mathematically gifted students, thus, are encouraged to go further in their explorations and discoveries based on analogical reasoning.

Sriraman (2003) observed that gifted student thinking behaviours including creative problem solving, formalization, and so on corresponded to those of mathematicians. Lee (2005) found that gifted students eagerly seek advance to higher level of reasoning through reflective thinking or reflection on their earlier thoughts and reasoning. Similarly, mathematically gifted student behaviour has been characterized as being faster; i.e., they are able to efficiently visualize, generalize, simplify, abstract, and grasp the meaning and structure of a problem in a much shorter time period than their peer groups (Heid, 1983; Sriraman, 2004). However, there have been relatively few studies on the analogical reasoning of the gifted.

One of the major challenges in gifted education is to develop an educational program that conforms to the characteristics and needs of gifted students. To date related studies have revealed, at least three requirements that must be met: access to advanced mathematical content (Johnson & Sher, 1997); exposure to challenging mathematics problems (Johnson, 1993); and opportunity to develop creative thinking (Sheffield, 1999). These three requirements were considered in the development of an educational program for the gifted that incorporates analogical reasoning. The objective of this study was to obtain detailed

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information on the way mathematically gifted students utilize analogical reasoning.

THEORETICAL FRAMEWORK

Rota and Palombi (1997) describe mathematician behaviour as the systematic concealing of analogical trains of thought, the authentic life of mathematics, by enrapturing discoveries. Poincaré points out that mathematical facts worthy of study are those that, by their analogy with other facts, are capable of leading one to knowledge of a mathematical law, just as experimental facts lead to knowledge of a physical law. More explicitly, Atiyah mentions that mathematics is the science of analogy, and he argues that finding analogies between different phenomena and developing techniques to exploit these analogies is the basic mathematical approach to the physical world (cited in Corfield, 2003: 81-82). Analogies, thus, can be seen as cognitive aids to the discovery and learning of mathematics.

Polya (1962) explains analogy as a kind of similarity. He claims that the essential difference between analogy and other kinds of similarity lies in the intention of the thinker. If one intends to reduce several aspects shared by objects in order to definite the concepts that make the objects similar, one clarifies the analogy. Polya explains his rational by reasoning that a triangle can be analogous to a pyramid: a triangle is obtained if all points of the segment are connected to a point outside the line of the segment, and a pyramid is obtained if all points of the polygon are connected to a point outside the plane of the polygon. Using this same reasoning, he shows why a prism can be regarded an analogue of a parallelogram.

Piaget (1952) claims that there exists spontaneous functioning of the schemata of displacement by analogy and this analogy entails imagination of new combinations. Analogy is, thus, an instrument that differentiates initial schema by assimilation. According to Piaget, there are two levels of relations that form the solution of analogy problems or analogical reasoning. Lower order relations are simpler ones that are generated between near or closely paired concepts. Higher order relations, in contrast, are those produced between more distant or removed concepts.

METHOD

To investigate the use of analogical reasoning by mathematically gifted students, this research paper intentionally samples appropriate cases, collects data via observation, and performs an in-depth analysis of the data as suggested by Strauss and Corbin (1990). The subjects of this research are two 12-year-old elementary 6th graders (E1, E2) and two 14-year-old junior-high 8th graders – all of whom are receiving education at an attached academy for the gifted at a university. E1 and M1 excel in algebra while the other two show greater aptitude in geometry, as determined from their 12-month-long education at the attached academy prior to this study.

A 9-hour long educational program (3 units, each composed of 3 hours) is provided to the gifted students. Each session with the elementary school students is conducted separately from the middle school students, and for each student one research assistant is assigned to conduct concentrative observation on and an interview with the student. This approach facilitates observation of the ways in which gifted students might differ from each other regarding: (1) the kinds and aspects of objects featured in gifted students' analogical reasoning, (2) the ways in which gifted students compare objects and find correspondences, and (3) the depth of gifted students' mathematical understanding as it emerges in the formulation and justification of an assumption by analogy. To ensure reliability, all utterances by students and interactions with interviewers are audio-/video-taped.

The tasks used in the nine-hour educational program are detailed below.

- [Task 1] Which geometric figure can be regarded an analogue of a triangle? Explain your answer.
  [Task 2] A triangle and a tetrahedron can be regarded analogous. How is this possible? Explain your answer.
- [Task 3] Make conjectures on a tetrahedron based on your knowledge of a triangle and justify your answers.

The above detailed tasks were developed with the objective of requiring students, individually, to discover possible mathematical conjectures on a tetrahedron and justify the conjectures using analogy. Hence, the classical form of an analogy problem, i.e., "A:B::C:?" in which three components of the problem are given and the fourth is to be determined, was modified to meet study objectives. The first task asks gifted students to find analogous objects without any given information about object relations. In other words, students attempted the problem "A:?::?:?". This modification was made to investigate whether gifted students possess innate analogy reasoning. The second task requires students to postulate relations in order to clarify the given analogy "A:?::C:?". By attempting this problem, individual divergent abilities surface as they compare two figures and make correspondent properties, which complete analogy. The final task differs from the classical analogy problem form in that it requires gifted students to formulate whole conjectures by analogy based on knowledge rather than simply determining the fourth component in an analogy problem. These modifications meet recommendations stipulated by prior research on gifted education (Johnson, 1993; Sheffield, 1999).

RESULTS

All four students actively participated in each of the task solving processes. Each student revealed quite different analogical reasoning, and table 1 highlights the kinds and characteristic aspects of each gifted students' analogical reasoning.

Task	E1	E2	M1	M2
1 (A:?::?:?)	Polygon Polyhedron Tetrahedron (elements)	Polygon Tetrahedron (feature)	Polygon Pyramid Tetrahedron (elements)	Tetrahedron (feature)
2 (A:?::C:?)	Role Feature	Property Relation	Feature	Property Relation
3 (A:?::C:?)	The sum of interior angles Number of sides, faces	The sum of interior angles Inscribed Face angle	The sum of interior angles Inscribed Circumscribed	The sum of interior angles Inscribed Face angle Solid angle

Table 1: Emerged analogical reasoning

Interestingly, all students, except M2, initiated the first task by drawing and analysing a triangle and various kinds of polygons. E1 and M1, both of whom are strong in algebra, investigated triangles and polygons by focusing on elements such as angles, sides, and faces. E2 and M2 who were identified as being strong in geometry focused on features of a triangle. They describe a triangle as "a fundamental figure by which all kinds of polygon can be made" (E2) and "a figure with the smallest number of sides in a plane" (M2). When postulating and justifying conjectures on a tetrahedron, each of the four students spent a considerable amount of time looking at the correspondent property to "the sum of the measures of an arbitrary triangle's interior angles is π."

Evaluation of Analogies

At the onset of the study programme, the instructor briefly explained the relationship "dog : bark :: cat : meow" (Alexander et al., 1997) to ensure students understood analogy. E1 found that a triangle, irrespective of its shape, has several unique elements such as vertices, angles, and edges. Quickly realizing that a polygon also has these same elements, E1 concluded that any polygon is analogous to a triangle. E1 quickly adopted his generalization to a polyhedron with the same reasoning. E2 and M1 seemed to proceed along the same route as E1. They explained that any polygon can be analogous to a triangle because of the similar expression for the sum of the measures of its interior angles and the same summation for the sum of the measures of its exterior angles.

After a while, all students deemed those analogies unproductive. The following is a part of a conversation between E2 and the interviewer.

E2: This is so boring! What can I do make it more exciting?

Interviewer: Are you not satisfied with your constructs?

E2: Not at all. But there must be something more important, something I am overlooking. My analogy just isn't that impressive.

Interviewer: You want to find something notable and interesting?

E2: Yes, I ought to find another example for the analogy.

While three of the students eventually turned to polyhedrons, which led to the consideration of a tetrahedron, M2 focused on tetrahedrons from the very beginning. The strong interest in tetrahedrons finally resulted in students making an analogy between a triangle and a tetrahedron. This observation provides evidence that gifted students are aware of the use of meta-cognition. That is to say, they can question and evaluate their own analogies, which leads to useful mathematical discoveries (Kramarski & Mevarech, 2003). The students are unlikely to have considered solid geometry or a tetrahedron as a correspondent object of a plane geometry or a triangle if they had been satisfied with initial analogies. They most likely would have stopped after making preliminary analogies.

M2, in particular, did not ponder long the analogy of a polygon or a polyhedron to a triangle. He quickly began focusing on a tetrahedron and spent most of his time pondering the relation or structure of the analogy he needed to make without producing much output. In the interview, he was asked why he decided to proceed immediately with an analogy attempt to a tetrahedron unlike other students. His response is detailed below.

M2: It just came to mind! A tetrahedron is very similar to a triangle indeed.

Interviewer: Why not consider a quadrangle? It's similar to a triangle, isn't it?

M2: Yes, in a sense. However, I think I need to change this (pointing the word "triangle" written by him) to something very similar and very different.

Interviewer: Why does it have to be different?

M2: I don't know but I think it should be; otherwise, nobody would welcome my idea.

According to his answer on the necessity of being different, he is seen evaluating his potential analogies of a triangle mentally. Therefore, even without a single explicit trial, he came to the idea of a tetrahedron. This finding suggests gifted students are willing to review their approaches to and results of finding analogous figures of a triangle, which easily leads to discoveries of useful analogy.

Once a similarity between a triangle and a tetrahedron had been established, students centred on describing the roles or features of the two figures in plane geometry and solid geometry. E1 and M1 wrote: “a triangle is the basis of all polygons and a tetrahedron is the basis of all polyhedrons. Both figures can be considered a starting point and the simplest among figures of each geometry.” In other words, once E2 and M2 completed a list of all the properties of a triangle and a tetrahedron, they tried to find a correspondent property for each of the listed properties.

Space of Knowledge and Thinking Strategies

According to Alexander et al. (1997), each turn of an analogy problem should involve encoding, and inferring takes place when a relationship is constructed between the first pairing in a stated problem to make analogy. During the second and third task, the four students’ space of knowledge on a triangle was significantly different as shown in table 2.

Space of Knowledge	E1	E2	M1	M2
The sum of the measures of an arbitrary triangle’s interior or exterior angles is π.	○	○	○	○
The bisectors of the angles of a triangle intersect at a point that is equidistant from the three sides of the triangle.	○	X	○	○
The perpendicular bisectors of the sides of a triangle intersect at a point that is equidistant from the vertices of the triangle.	○	X	○	○
The medians of a triangle are concurrent	X	X	○	○

Table 2: Difference of Space of Knowledge

The preliminary assumption, before the onset of this study, was that students’ space of knowledge is the main contributor to their analogical reasoning, especially at the encoding and inferring stages. E2, however, performed very well despite a relatively smaller space of knowledge on a triangle than E1. Also, M1 and M2 showed considerable difference in analogy creation even though their spaces of knowledge appeared very similar. E1 and M1 did not clarify analogy using knowledge of incentre and circumcentre or centroid, whereas E2 and M2 did and as a result reached a rich analogy.

The most impressive difference occurred when E2 and M2 focused on face angles and the sum of the measures of those angles in a tetrahedron. They discovered that the sum of the measures of an arbitrary tetrahedron's face angles is invariant identical to the sum of the measures of a triangle's interior angles. M2 realizing the need to define the angle made by three faces in the corner of a tetrahedron concentrated on the relationship between a solid angle and the sum of the measures of face angles. He made the conjecture: "the sum of solid angles of a tetrahedron is constant akin to the sum of the measures of face angles of a triangle." In contrast, E1 and M1 did not look at face angles. They focused their attention on the measure of solid angles of a tetrahedron. Both ultimately asked the instructor for solution hints and in the end gave up on the third task.

CONCLUSION

As Sheffield (1999) points out, talented students should be inspired to think like mathematicians. Participants in this study appeared to experience the deep thinking that is necessary to solve problems made with analogies, a process equivalent to the one that mathematicians undertake. The subjects had to reflect on prior knowledge and develop new concepts such as face angle and solid angle based on analogical reasoning. Students with excellence in algebra (E1, M1) had relatively more difficulty clarifying higher-order analogy than students deemed good at geometry (E2, M2). Quality in the justification of the conjectures by analogies also differed between the two groups. This finding suggests a possibility that analogical reasoning is connected to student learning preference or capacity in content areas. All subjects, however, were found adept at making meaningful analogues of a triangle since they all made use of meta-cognition when searching relations for analogies. In the future, methodologies including the development of tasks and teaching settings, measures to evaluate the depth of mathematic exploration through analogy, and research on how to promote education related to analogy for gifted students will enhance gifted student mathematics education.

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