THE DEVELOPMENT OF ALGEBRAIC REASONING IN A WHOLE-CLASS SETTING

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Generalization is a powerful vehicle for the development of mathematical insight by primary school pupils. However the implementation of this in classrooms is challenging for teachers as pupils often focus on superficial aspects of patterns. In this paper there is a description of a whole class discussion that shows that generalization and justification are closely aligned. In this regard, there is a need for teachers to engage in pedagogical press and for students to attend to the functional relationship between variables rather than pattern finding in single variable data.

INTRODUCTION

Over the past decade the teaching and learning of algebraic reasoning has been a focus of research and reform efforts. Among the arguments posited by Kaput and Blanton (2001) for the inclusion of algebra in primary grades are access to powerful ideas and the addition of a new level of depth and coherence to elementary mathematics. Algebra now appears in the curricular documents of many countries with the recommendation that it be taught throughout the primary school (D.E.S./NCCA, 1999; DfES, 2001; NCTM, 2000). The extent to which students are given the opportunity to access the powerful ideas of algebra is questionable, however. In Ireland, reviews of the implementation of the 1999 Mathematics Curriculum suggest that emphasis remains on the strand of number and on lower-order thinking skills (Shiel, Surgenor, Close, & Millar, 2006). From an examination of text-books which are used prolifically for the teaching of mathematics (ibid), the focus in algebra remains on symbol manipulation and pattern generation with little opportunity for generalization and justification. As whole-class discussion can be fertile ground for the development of higher-order mathematical thinking (O'Connor, 2001) it may be a means of addressing this issue.

LITERATURE REVIEW

In early years' classrooms, the emphasis in algebra is usually on the exploration of simple repeating and growing patterns. As any variation usually occurs within the pattern itself, the focus is on single variational thinking. As pupils move through the primary school system greater emphasis is placed on the formation of functional relationships and the generalization of patterns. Functional thinking is described by Warren, Cooper and Lamb (2006) as a focus on the relationship between two or more varying quantities. Although young children are capable of thinking functionally (Blanton & Kaput, 2004; Warren, 2005; Warren et al., 2006), there is evidence that

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they focus on pattern spotting in one data set rather than on the relationship between the pattern and its position throughout the primary years. Warren (2005) suggests that that this is the case either because single variational thinking is cognitively easier for children or is so engrained from school experience that there is a tendency to revert to it. Lannin, Barker and Townsend (2006) outlined a continuum of generalization strategies:

- Recursive: The students describes a relationship that occurs in the situation between consecutive values of the independent variable
- Chunking: the student builds on a recursive pattern by building a unit onto known values of the desired attribute
- Whole-object: the student uses a portion as a unit to construct a larger unit using multiples of the unit.
- Explicit: a rule is constructed that allows for immediate calculation of any output value given a particular input value.

The framework developed by Lannin et al (2006) was based on a teaching experiment with pairs of fifth-grade students who engaged in the solution of linear generalization problems. They found that students' choice of strategy was influenced by social, cognitive and task factors. In particular, student desire for efficiency played an important role in the method they used. The use of close input values (e.g. consecutive input values) seemed to invoke the recursive strategy while input values that were multiples of prior known values were conducive to the use of the whole-object strategy. In general it was found the students who had poor visual imagery of the problem often used the recursive or whole-object strategies incorrectly. Indeed, the tendency to use the whole object strategy erroneously (to assume, for example, that $f(100) = 5f(20)$) is commonplace among primary and secondary students (Stacey, 1989).

Anthony and Hunter (2008) drew on the framework above in their research with a class of students aged 9 – 11 years. Similar to Lannin et al., they found that social factors impinged on choice of method. The sharing of strategies by students in small groups appeared to encourage some but not all students to consider more efficient strategies. Generally speaking, students seemed to opt for the recursive strategy before risking alternative strategies. Teacher pedagogical press was another factor that appeared to support the development of more flexible, efficient strategies. The need for generalization and justification to be strongly linked is well recognised (Lannin, 2005; Lannin et al., 2006; Rowland, 1999) and is the subject of recent research (Ellis, 2007). The way this link might be forged through whole-class discussion is the focus of this paper.

BACKGROUND

The aim of my research is to investigate the factors that contribute to the development of mathematical insight by primary school pupils. The theoretical framework of the study is derived from the sociocultural perspective. The methodology is that of 'teaching experiment' which was developed by Cobb (2000) in the context of the emergent perspective and in which students' mathematical development is analysed in the social context of the classroom. I taught mathematics to a class of thirty-one pupils (seven girls and twenty-four boys) aged 9 - 10 years on a total of twenty-seven occasions over a six-month period. Many lessons took place over two or three consecutive days, each period lasting forty to fifty minutes. The school is situated in Ireland in an area of middle socio-economic status. All phases of the lesson were audiotaped and detailed field notes were written. When children were working in pairs, audio tape recorders were distributed around the room. Each pupil maintained a reflective diary. Follow-up interviews were held with students who had shown some evidence of reaching new understandings over the course of a lesson.

As it had emerged from data collected at the pilot stage of the project that generalization was conducive to 'insight', I sought to include an element of it in each lesson taught. In keeping with grounded theory methods, data collection and analysis occurred simultaneously. The data were analysed using a data reduction approach. Initially a descriptive account of the events for each session was constructed. I used these descriptive accounts to identify emerging themes and to create codes and categories (Charmaz, 1995; Goetz & LeCompte, 1984). A comparative analysis of lessons was also undertaken (Glaser & Strauss, 1967). Connections to previous literature and the new ideas that arose were documented.

The problem that is the focus of this paper was entitled. 'Friendship Notes' and involved finding the number of sheets of paper required if notes were to be sent by each member of a group to all other members, i.e., if there were ten pupils in a class the number of notes required would be ninety as each child would send notes to nine of his or her class mates. The solution generalizes to n(n-1) and as such is a quadratic[1] rather than a linear problem. The lesson took place over three consecutive days but transcripts from the plenary sessions that took place on the first day only will be considered here.

FINDINGS

In introducing the problem I clarified the notion of a friendship note (a friendly note written by each pupil in a group to all others) and the fact that the teacher had to order a set of blank notes for this purpose. I first asked the pupils to consider a situation where there was only one child in the group and then two. The focus at this stage was on clarification of the problem conditions, i.e., pupil would send one and only note to each member of the group and not to him or herself. When three members of the class were being considered in general a count strategy was used, although erroneously as was the case below with Aidan and Catherine[2], e.g.,

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[1] In the generalized form of a linear function, 1 is the highest power. In a quadratic function, 2 is the highest power.

[2] The following transcript conventions are used: T.D.: the researcher/teacher (myself); Ch: a child whose name I was unable to identify in recordings; Chn: A group of children speaking in unison;…: a hesitation or short pause; […]: a

77 Aidan: Three.

78 T.D.: Right you think three. Why do you think three?

79 Aidan: Because they have to get two, and then there's another one so like they get one each, so three.

80 T.D.: Three. Ok

81 Ch: Ah Miss!

82 Catherine: It would be nine.

85 T.D.: Why do you think nine, Catherine?

86 Catherine: Cos Chris has to send one to Charlie and Sinead and then Charlie will send one to Chris and Sinead and …

87 T.D.: And would that, would that be nine?

88 Catherine: Yeah.

89 T.D.: You think it would be nine, ok. Anybody else think different?

92 Myles: Six.

93 T.D.: Why do you think six?

94 Myles: Because each person has to send two notes, Chris sends one to Charlie and Sinead and then Sinead will have to send one to Charlie and Chris and Charlie will have to send one to Chris and Sinead.

Aidan, it seems, interpreted that the problem concerned receipt of one note each. It is possible that Catherine supposed nine because she had not given any thought to the non-reflexive nature of the activity. The problem was then enacted by the three pupils in much the same way as suggested by Myles (turn 94). A list was made on the blackboard showing number of children alongside number of notes. When a fourth pupil joined the group, Desmond used a recursive strategy incorrectly:

141 Desmond: Ten.

142 T.D.: Ten, you think ten. Why do you think ten?

143 Desmond: Because it is going like two - two, six, so it is going up four.

144 T.D.: Ok so you are looking here like this – two, six, you are looking at the …the list on the blackboard, so you think it is going up four each time. So that could make complete sense…

He is assuming, possibly from prior experience, that the difference between output values is constant. Brenda restated this idea:

145 Brenda: Eh, ten, cos on that side it has like a pattern.

Other students began to use the multiplicative relationship between both variables:

151 Myles: Em twelve.

152 T.D.: Why do you think twelve?

153 Myles: Because it would kind of be like four threes and four threes equals twelve.

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pause longer than three seconds; ( ): inaudible speech; [ ]: lines omitted from transcript because they are extraneous to the substantive content of the lesson.

154 T.D.: And why do you think it's four threes?

155 Myles: Cos eh one person would have to give three notes to the other three people.

156 T.D.: You think twelve. What do you think?

157 Catherine: Twelve.

158 T.D.: Why do you think twelve, Catherine?

159 Catherine: Because each person has to send three notes.

Catherine it seems has moved from a counting strategy (see turn 86) to consideration of the relationship between the variables. When the group size was extended to five, a few pupils used the multiplicative relationship but Dermot thought the answer might be fifteen 'cos four would need twelve and add another three' ('chunking' strategy). Finbar then made the following contribution:

268 Finbar: It's like eh 1, em it actually kind of starts on two, it goes two then like count up in twos but it skips four, goes on to six and then that it skips eight and then it goes on to twelve but then on the fives it actually just em skips em eh fourteen, sixteen and eighteen, just goes on to twenty.

269 Chn: Huah[3]>huah>huah>please

270 T.D.: Right … yes.

271 Liam: It goes two, four, six, eight, ten.

272 T.D.: Does it go two, four, six, eight, ten?

273 Liam: No it doesn't go up, it's cos it's, cos zero plus two, two plus four equals six, six plus six equals twelve, twelve plus eight equals twenty.

Finbar has seen an additive pattern on the right hand side of the table and Liam expanded on his idea. Alex then saw the functional relationship between the numbers on the right-hand side and the left-hand side of the table. However his attention remained on the pattern of that relationship:

279 Alex: Em well on the board cos em 1 goes into zero one and then two goes into two once and then three goes into three (sic) twice, four goes into twelve three times and then five goes into twenty four times.

None of the three, Finbar, Liam or Alex, was able to use the pattern they found to predict answers for higher numbers, possibly because they had not given consideration to the structure of the problem. I distributed the worksheets in which pupils were asked to consider the number required for six to ten children. They worked in self-selecting pairs or triads on the worksheets. During the plenary session that followed the completion of these sheets, most pupils seemed to be using a multiplicative strategy to find solutions between one and ten and were able to justify their reasoning, e.g., in response to the number of notes required by six children the following interchange occurred:

[3] 'Huah' is used to indicate the sound made when there is a sharp intake of breath; it suggests that a child is keen to make a contribution to discussion.

298 Alex: Thirty.

299 T.D.: Right, now you have got to say why, why do you think it's thirty?

300 Alex: Because fi, cos you don't write one to yourself so five by six. [ ]

305 T.D.: …What would it be for seven? Hugh?

306 Hugh: Eh forty-two.

307 T.D.: Forty-two, can you explain why?

308 Hugh: Cos seven sixes is forty two.

309 T.D.: And where are you getting seven sixes from?

310 Hugh: Em cos there's seven pupils, then they each have to send six.

Alan made reference to the multiplicative relationship and justified his reasoning as a result of teacher press:

321 T.D.: Where are you getting the seventy-two from?

322 Alan: Em nine multiplied by eight is seventy two.

323 T.D.: And why is it nine eights?

324 Alan: Em, I don't know. I just found a pattern like that.

325 T.D.: You found … what kind of a pattern did you find?

326 Alan: Em if it was six pupils it would be six multiplied by five.

327 T.D.: And so why is it six multiplied by five if there were six pupils … Why? []

330 Alan: Cos they can't give one to themselves.

331 T.D.: Ok, so if there were ten how many would there be?

332 Alan: Em, ninety.

When discussion moved on to consideration of the number of blank notes needed for twenty pupils, Rory used a whole object strategy but undercounted. He decided on one hundred and eighty on the basis that ninety notes would be required for ten pupils:

334 Rory: One hundred and eighty.

335 T.D.: Where are you getting one hundred and eighty from?

336 Rory: Eh, well I just eh added ten and ninety and then I got a hundred and I added another eighty and I got eh one hundred and eighty.

In considering the solutions for inputs of twenty and later thirty, many pupils did apply the multiplicative relationship but became overly concerned with calculation, e.g.,

389 Colin: Eh you can take away the zero from the thirty …

390 T.D.: Hm, hm.

391 Colin: and take away one from the other thirty and then you have twenty-nine so you take away, you'll have to put up twenty-nine then you'll add three and two and you get five and then you add, you … wait, it's …

392	T.D.:	What do you think on the calculator? If you had a calculator now, you are getting into funny numbers now, hard numbers, what would you multiply on the calculator? Colin?
393	Colin:	Thirty by twenty-nine.
394	T.D.:	Thirty by twenty-nine. Do you agree with that, David?
395	David:	Yes.
396	T.D.:	Why?
397	David:	Eh because em if there are thirty in the class you wouldn't be giving themselves notes ()

The lesson ended soon after and was revisited on the next day when there was further focus on the generalized pattern.

CONCLUDING REMARKS

Initially most of this group of students used a count strategy which was verified by pupil modelling of the situation. As they became accustomed to the problem conditions more of them gave consideration to the multiplicative relationship between the variables. As results were listed on the blackboard, a few pupils began to use a recursive strategy to find the solution for an input value of four. As suggested by Lannin at al (2006) and Anthony et al (2008), this may have been due to the fact that input values were consecutive - it is possible that the use of random rather than consecutive input values would help pupils to focus more attention on the functional relationship. At the conclusion of the lesson some class members seemed to have developed an explicit rule (thirty by twenty-nine) - whole class discussion and teacher press for justification played important roles in the development of this rule. Similar to the findings of Warren et al. (2006), attention shifted from pattern identification to computation when large numbers were introduced. Also the erroneous use of the whole object strategy (see turn 336) was a prevalent feature in many lessons in this teaching experiment and warrants further investigation. Finally although the recursive strategy is not necessarily helpful for the development of a generalization, the pupils appeared very excited when they found a pattern (see turn 269) – how to harness this excitement into a deeper appreciation of algebraic structures remains a challenge for mathematics education researchers and teachers.

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