DIFFERENT PERCEPTIONS OF INVARIANTS AND GENERALITY OF PROOF IN DYNAMIC GEOMETRY

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Literature on research in dynamic geometry environments (DGEs) addresses the importance of the perception of invariants in open problem investigations. In this paper, in order to analyse students' processes during exploration, conjecturing, and proving in DGEs, we introduce a framework that distinguishes different types of invariants. Students' interpretation of these invariants seems to be strongly rooted in the processes of their discovery more than in the generality of theorems and proofs.

INTRODUCTION

The importance to mathematicians of visualization is widely recognized, and recently the appearance of Dynamic Geometry Environments (DGEs) has led educators to reconsider the issue of imagery in mathematics education (for example, Goldenberg, 1995; Presmeg, 2006). Certainly DGEs have revolutionized the approach to understanding the complex relationship between images and concepts in Geometry education. In particular, DGEs have been expected to enhance geometrical reasoning in problem solving, promoting visual exploration and discovery (Dreyfus, 1993; Goldenberg, 1995; Goldenberg & Cuoco, 1998). Laborde, speaking of a specific DGE, Cabri géomètre, states (Laborde, 1993, p. 56):

> "The nature of the *graphical experiment* is entirely new because it entails movement. The movement produced by the drag mode is the way of externalising the set of relations defining a figure. The novelty here is that *the variability inherent in a figure is expressed in graphical means of representation* and not only in language. A *further dimension is added to the graphical space as a medium of geometry:* the movement."

However, the complexity between spatial-graphical and geometrical aspects that is intrinsic of geometrical reasoning, cannot magically dissolve. If one hand DGEs seem to foster the development of a link between spatial-graphical and geometrical aspects; on the other hand, they do not seem to foster achievement of the theoretical control over the relationship between purely spatial-graphical properties and theoretical properties of the figures represented (Duval, 1993; Laborde, 1993).

The intrinsic complexity of geometrical reasoning is nicely expressed by the notion of *figural concepts* introduced by Fischbein (1993). Geometrical figures are mental entities that simultaneously possess both conceptual properties (general, abstract relations, deducible in the Euclidean theory) and figural properties (shape, position, magnitude). In solving geometrical tasks, the interaction between figural and conceptual components of external representations (drawings) may explain productive reasoning leading to correct solutions, but it may also explain mistakes

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and difficulties that could be due to an incomplete fusion between the two components (see, for example Mariotti, 1993; Mariotti & Fischbein, 1997). In the case of tasks to be accomplished in a DGE, figures, that is drawings produced through a sequence of commands chosen by the user, are drawings with their own intrinsic logic. The geometric properties defined by the commands control the appearance of invariants under dragging, so the intrinsic logical dependency among the elements of the dynamic figure may affect the interplay between the figural and the conceptual components involved in the solution of a task.

This study is part of a greater project in which we analyse some cognitive processes that students activate when solving open problems requiring the formulation of a conjecture in a DGE. The study involves students in grades 9, 10, and 11 who have been using a DGE in the classroom for at least one year. During clinical interviews the students work in pairs or singularly while being audio and video recorded. The paper aims to propose a classification of different types of invariants that seems to be powerful for analysing and explaining some difficulties encountered by the students.

GEOMETRICAL INVARIANTS

In the literature on research in DGEs, the terms "invariants," "geometrical invariants," "invariant properties" of a figure have been used to refer to certain properties (e.g. Yerushalmy et al., 1993; Goldenberg et al., 1998; Hadas et al., 2000) that are maintained when some transformations on the figure are performed. As Laborde describes it:

> "A geometric property is an invariant satisfied by a variable object as soon as this object varies in a set of objects satisfying some common conditions." (Laborde, 2005, p. 22).

In the work we have so far accomplished for the study, we have noticed recurring student-behaviours that are not consistent with a correct mathematical interpretation of the situation. As noticed by other researchers (for ex. Laborde, 2005), we also are finding that students encounter difficulties in dealing with dependency relationships of points and interpreting invariants of a figure. In order to analyze students' behaviours, we find it useful to distinguish different notions of invariance. In order to define them, in a more precise manner, we first focus on the dragging of points, and make the distinction between base-points and constructed points. A base point is a point constructed freely, or semi-freely (for example, a point chosen on a circumference, which can therefore only be dragged along the circumference itself) on the screen, upon which other objects of the construction depend. A base point is also a free point, in that it can be dragged anywhere on the screen (or freely along the curve it is linked to). On the other hand, dependent points are points built as the intersection of constructed objects, and consequently they cannot be dragged directly.

We call *construction-invariant* a geometrical property of the figure which is true for any choice of the base-points. In Cabri an invariant of the construction is a property that is maintained for dragging of any base-point (which is also free) of the figure. It is useful to consider the set of all construction-invariants, which we will call *I*.

We may also consider a geometrical property that is true for any choice of one particular base-point of the construction, while the other base-points are fixed. In fact, only one point at a time can be dragged, so students actually perceive properties as being invariant for the dragging of the specific point they are dealing with at the moment. In this case we have a *point-invariant*. If the particular base-point considered is P, we will call such invariant a *P-invariant*. It is useful to consider the set of all P-invariants, which we will call I_P. It is clear that I is contained in I_P for every base-point P of the construction.

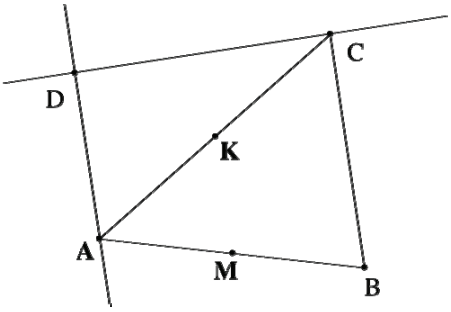

Fig 1: A, M, K are base points and free in this construction; "ABCD is a right trapezoid" is a construction-invariant; "the length of BC" is an A-invariant.

For example consider the following construction. Let A, M, K be three base-points, and construct B as the symmetric point of A with respect to M, and construct C as the symmetric point of A with respect to K. Construct the parallel line l to BC through A, and the perpendicular line r to l through C. Let D be the intersection of l and r.

It is easy to prove that ABCD is a right trapezoid for any choice of A, M, K. By construction DA lies on l, which is parallel to BC, and CD lies on r, which is perpendicular to l. Consequently, r is also perpendicular to BC (for a known theorem of Euclidean geometry). Therefore the fact that ABCD is a right trapezoid is a *construction-invariant*. Moreover, the fact that BC is twice MK and parallel to it is also a *construction-invariant*. This is because the triangles AMK and ABC are similar with ratio of proportionality 1:2 (since AC is twice AK, and AB twice AM). However, the fact that the length of BC is constant is not a *construction-invariant* because, for example, choosing a different M (or dragging M) leads to a variation of it. Instead, the length of BC is an *A-invariant*, because for any choice of A (or movement of A through dragging), MK is fixed and therefore the direction and length of BC are constant (even if the points B and C change as a consequence of the new choice of A).

We will now show how the notions of construction-invariants and P-invariants, together with that of figural concept, can be efficiently used in our analysis. In the following we consider the representative case of a pair of students engaged in the solution of an open problem.

ANALYSIS OF A TRANSCRIPT: THE CASE OF GIULIO AND FEDERICO

Below are some excerpts from the transcript of an interview of two students, Giulio and Federico. The task is based on the configuration of the example above. After accomplishing the construction, the students are asked the following: "As points A,

M, K vary, formulate conjectures on the types of quadrilaterals that ABCD can become, trying to describe *all* the ways in which it is possible to obtain a certain quadrilateral." Giulio and Federico are in the second year at an Italian high school (grade 10) and have used dynamic geometry in class during their previous year.

The two students have been looking at the screen while Federico has been dragging point A randomly. In the excerpts below, I indicates the researcher.

Excerpt 1

1	F:	Good, so we can say that the quadrilateral, as A varies, uh, we always have a trapezoid.
2	G:	…a right trapezoid.
3	I:	It's a trapezoid.
4	G:	a right one.
5	F:	a right one! Yes, it is a right trapezoid.
6	I:	Ok, it's even a right one.
7	F:	So, dragging A…it's a right triangle…yes.

While Federico is dragging point A, the two students observe that ABCD is "always" a right trapezoid. Therefore, we can say that they conjecture that the property "ABCD is a right trapezoid" is an *A-invariant*. This also emerges from Federico's words every time he formulates the conjecture ("as A varies" (1), "dragging A" (7)).

In the following excerpt the students are involved in the production of the proof of the conjecture.

Excerpt 2

13	F:	Yes, let's prove it. Let's prove this one [conjecture] right away. So, by hypothesis we have that CD is perpendicular to AD.
[…]		
16	G:	Yes, and we also have that by hypothesis AD, since it lies on line *l*, is parallel to CB.
17	F:	Yes…
18	G:	uh, yes, because it's written [referring to the text of the problem].
19	F:	Yes.
20	G:	Construct line *l* parallel [G rereads the text to F] to BC.
21	F:	Good, so…
22	G:	So we know that …[speaking together]
23	F:	Well, we have that AD e BC are parallel *exactly* by hypotheses.
24	G:	So ABCD is a trapezoid….immediately.
25	F:	Yes and ABCD is a trapezoid, exactly! Then we already said this, AD [indicating with the pointer] is perpendicular to CD, but AD is parallel to BC, so AD is perpendicular to DC, but BC is also perpendicular to DC.

26	I:	Yes…
27	F:	So, it's a theorem!
28	I:	you don't need to prove the theorem [smiles]
29	F:	uh, we did it [in class].
30	I:	What is it that the theorem says?
31	F:	Uh, that if there are two parallel lines…
32	I:	Yes…
33	F:	uh, if we draw a perpendicular to one of them…
34	G:	uh, there are alternate interior angles…in the end
35	F:	Exactly.
36	G:	Because these two have to be supplementary.
37	I:	there we go, supplementary. I agree.
38	G:	Yes, therefore…
39	F:	uh, it has to be that if one is 90, the other, too, has to obviously be 90. Ah, so then it is proved, so it is a right trapezoid.
40	I:	Alright.

Giulio and Federico prove their conjecture correctly without making use of the dragging tool and, in fact, while the proof is produced, the figure remains static on the screen. Moreover, no reference to a particular choice of the base-points appears in the proof and its generality is assured by the theory of Euclidean Geometry. Therefore, the students have proved that "for any base points A, M, K ABCD is a right trapezoid". According to the notions of invariants proposed above, we can say that Giulio and Federico proved that the property "ABCD is a right trapezoid" is a *construction-invariant*. After the proof one would expect that the *A-invariant* should have changed its status for the students, becoming a *construction invariant*. However, from a cognitive point of view, the generality of the proof does not seem to have such effect on Federico's interpretation of the invariants, as the following excerpt shows.

Excerpt 3

41	F:	Now let's try something else…Let's do free dragging…
42	I:	Try to see if it can be something else…like other quadrilaterals.
43	F:	Still dragging A?
44	I:	Well, it says as A, M, K vary. So you can drag…
45	F:	So A, M, K [looking at them on the screen]…also M and K?
46	I:	Also M and K.
47	F:	Let's try…dragging M [F drags M freely] they all vary…yes, all the ...the sides…
48	I:	uh huh…
49	F:	Let's see…what can we say?…[he continues to drag M]…well, it seems to be again a trapezoid.

Federico proposes the same conjecture they have already proved as if it were a new property. In fact, even if this conjecture is now a mathematically proved proposition, we claim that for Federico this is actually a new conjecture (49: "It seems to be again a trapezoid"): according to our framework, he is saying that the property "ABCD is a trapezoid" is an *M-invariant*. We know that every *construction-invariant* is a *point-invariant* but, for Federico, this property does not seem to be a *construction-invariant*, even if a general proof has been constructed. For him, the property is only an *A-invariant* and therefore it might not be a *P-invariant* for other points P. For this reason the fact that it is an *M-invariant* is surprising to him. In the last part, Giulio's different interpretation of the invariant appears:

Excerpt 4

50	G:	Well, it is always *anyway* a trapezoid.
51	F:	Always a trapezoid, exactly.
52	G:	because by hypothesis, basically.
53	F:	It seems to always be…

Giulio is convinced that "ABCD is a trapezoid" is a *construction-invariant* ("it is always anyway a trapezoid"), therefore it is obvious for him that it is an *M-invariant*.

During the whole interaction, between Federico and Giulio there seems to be an underlying non-complete understanding that can be explained by the different way in which the two students treat the property "the quadrilateral is a right trapezoid." Federico treats it as an *A-invariant*, while Giulio treats it a *construction-invariant*. In this regard, we can notice the expressions that the two students use to talk about the conjecture: while Federico specifies that the quadrilateral is a trapezoid "as A varies" or "dragging A", Giulio never associates the claim of the fact with dragging point A (or any other), and in 50 he probably tries to underline his belief by adding "always *anyway*" [in Italian: "sempre *comunque*"] to his statement. Giulio seems to be convinced of this fact from the beginning of this sequence (after the initial dragging of point A that Federico does), even before he and Federico prove it.

Moreover, the interaction between the two students can also be explained well through the different interpretation that Federico and Giulio give of the observed invariant: their different interpretations of invariants lead to the astonishing (to us) exclamation of Federico ("It seems to be again a trapezoid") and to the, almost irritated, answers of Giulio ("It is always *anyway* [sempre *comunque*] a trapezoid").

CONCLUSIONS

As described elsewhere (Laborde 1998, p. 190) in the case of construction-tasks, the dynamic possibilities of Cabri may introduce a new level of complexity due to the possible variations of the given elements of the problem and their interrelations. Of course this consideration applies to other explorations. The previous examples show how such complexity emerges in the solution of an open problem and may affect the heuristic phase. The classification of invariants we have proposed seems to be a

useful tool to describe and explain such complexity. On one hand exploring by dragging may lead to noticing an invariant, but the geometric interpretation of the invariant may be strongly linked to the dragging process that lead to its discovery, thus it will be conceived as a point-invariant. On the other hand, the production of a proof may have different effects with respect to the generality of the geometric property related to the observed invariant. While Giulio seems to have grasped the generality of the theorem (the property for him becomes a construction- invariant), Federico does not. The link between the dragging process and the interpretation of the invariant is so strong that not even the production of a correct mathematical proof can induce Federico to change the status of the observed invariant. It may not be a chance that Giulio is the one who shows awareness that the invariant is independent of the dragging of any point, because he was not doing the dragging. It might be that the difference in the perception of the figure could depend on the type of control that each student has over the figure itself. Further research is needed to explore this idea.

Thus it seems that the basic claim that "spatial location and muscular movement in space are linked to variance and invariance, which lies at the heart of awareness of generality" (Mason & Heal, 1995, p. 298), has to be refined. The frame of Figural Concepts helps us to articulate the complexity of this process. Consider the geometrical interpretation of the observed invariant: the figural aspect is linked to the invariant spatial properties that are perceived under the constrains of the dragging experience. Thus it is not surprising to find out how persistent the effect of the dragging action might be. It is worth noting that the contribution of perception has to be considered in its complexity, "spatial location and muscular movement".

In line with findings of previous studies (for instance, Fischbein, 1982; Balacheff, 1988; Chazan, 1993) in a traditional paper and pencil setting, our students encounter difficulties in capturing the "generic" in a proof. According to our interpretation, once the proof is produced a new relationship between the figural and the conceptual component has to be elaborated. In the interpretation of the invariant, the point dependence of the property has to be overcome in order to achieve the generality stated by the proof. It may happen that such new elaboration is not accomplished and the generality of the property is not recognized. In terms of figural concepts there is a break between the figural and the conceptual aspect that needs to be recomposed.

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