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WARRANTS AS INDICATIONS OF REASONING PATTERNS IN SECONDARY MATHEMATICS CLASSES

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One crucial feature of mathematics as a discipline is its reliance on proof and deductive reasoning. If school mathematics is to allow students to engage in doing mathematics, the kinds of reasoning they encounter in their mathematics classrooms is important. As part of a study investigating the characteristics of argumentation and teachers' support for argumentation in secondary mathematics classes, we classified the warrants explicitly provided in arguments to examine what kinds of reasoning were accepted and supported in these classes. We found the most common warrants to be mathematical knowledge (theorems, definitions, results established previously in the class) and calculations. This was true across two different teachers' classes. However, in-depth analysis revealed differences in how and when these warrants were used in the classes.

BACKGROUND

The importance of reasoning in the study of mathematics and, indeed, in the development of life skills, cannot be overstated. In the United States, recent national policy documents have illustrated a continued focus on reasoning in K-12 mathematics. The National Council of Teachers of Mathematics' (NCTM) recent publication of the Focus in High School Mathematics series describes reasoning and sense making as the cornerstones of mathematical understanding for all high school students (e.g., NCTM, 2009). This set of publications builds on earlier research that suggests that students need to develop an understanding of mathematics as more than simply a collection of procedures (see, e.g., Cuoco, Goldenberg, & Mark, 1996; Kilpatrick, Swafford, & Findell, 2001). The release of the Common Core State Standards (Council of Chief State School Officers and the National Governors Association Center for Best Practices, 2010), which includes attention to practices such as making sense of problem situations, reasoning in multiple ways, and creating and critiquing arguments generated by others, brings more national attention to the importance of students learning to reason, communicate their reasoning, and critique the reasoning of others. Boaler, in several studies, has illustrated the importance of reasoning and sense making for students, not just in immediate mathematical performance, but also in career and later life decisions (Boaler, 2011).

Given the importance of reasoning in mathematics, it is important to find ways to examine what reasoning looks like in mathematics classes. One lens that provides insight into the reasoning practices of students and teachers is that of collective argumentation. Collective argumentation occurs in a mathematics class when the teacher and students work together to establish or reject mathematical claims. Yackel (2002) suggests that the teacher plays a particularly important role in facilitating collective argumentation. The mathematics education research literature contains multiple examples of teachers facilitating collective argumentation in ways that lead to student learning, exhibit students' reasoning, and allow students to make sense of mathematical ideas (e.g., Forman, Larreamendy-Joerns, Stein, & Brown, 1998; Krummheuer, 1995, 2007). This study builds on the extant literature on collective argumentation by using one part of Toulmin's (1958/2003) description of argumentation, the warrant, to provide insight into the kinds of ideas on which students and teachers base their reasoning.

FRAMEWORK

Much of the research involving collective argumentation in mathematics education draws on Toulmin's (1958/2003) description of the structure of an argument (see Figure 1 for the parts of an argument with brief descriptions of each). Toulmin argued that the structure of an argument is not dependent on the field in which it occurs, even though what is accepted as valid within an argument (in particular, data, warrants, and backings) is dependent on its field. These diagrams of arguments can be expanded to include sub-arguments (what Toulmin called preliminary arguments or lemmas) and can be coded to indicate the contributor(s) of each component (see Conner, 2008).

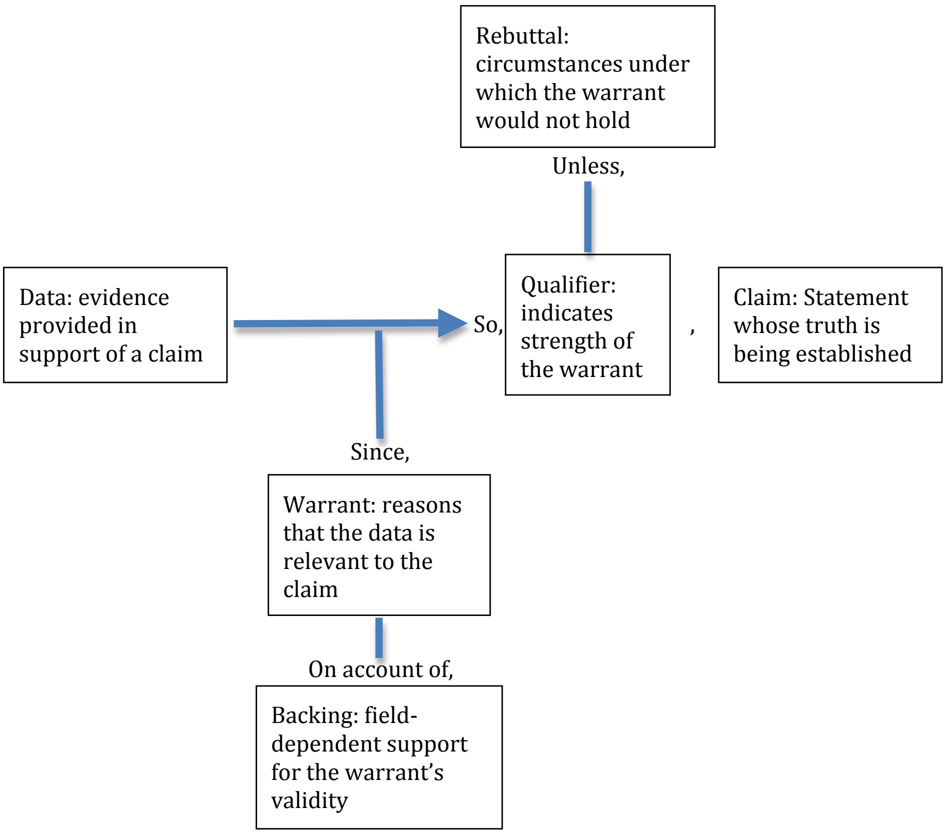

Figure 1: Argument structure (adapted from Toulmin, 1958/2003)

While arguments themselves give indications of the chains of reasoning that are used and/or accepted in a discussion, the warrants that are explicitly provided give insight into what the students and teachers are using as acceptable rationales in the classroom. Our focus on warrants is not unique: Inglis, Mejia-Ramos, and Simpson (2007) classified warrants used by their mathematically knowledgeable participants as inductive, structural-intuitive, and deductive, indicating the kinds of reasoning used in particular arguments. Nardi, Biza, and Zachariades (2011) also focused their analysis of teachers' reasoning on the warrants they provided, classifying their warrants according to a "range of influences (epistemological, pedagogical, curricular, professional, and personal)" (p. 4).

METHODS

As part of a larger study, we observed two student teachers in two different classes in the same school district as each taught a unit focused on geometry. Bridgett[1] was teaching an integrated and accelerated ninth grade class, and Kylee taught an on-level tenth grade geometry class. We video recorded each class, focusing on the teacher, in order to capture the teacher's support for collective argumentation. We transcribed each class session from the video recording, augmenting the transcripts with worksheets provided by the teachers and field notes from the observations. Our data corpus included video recordings, field notes, and supplementary materials from eight days of Bridgett's class and nine days of Kylee's class.

Our analysis began with identifying episodes of argumentation within each class day. Each major claim indicated the presence of an episode of argumentation. Often, an episode of argumentation included a main claim, the data, warrant, qualifier, and rebuttal for that claim, and multiple sub-arguments that supported the data and possibly the warrant for the main claim and/or other claims in the argument. Individuals and pairs of researchers within the research group diagrammed each argument, indicating with color who (the teacher, the students, or a combination of teacher and students) contributed each part of the argument (see figure 2 for an example of an argument). When a part of an argument (usually the warrant) was not explicitly spoken or written in the class, but it seemed clear that it was understood by at least part of the collective, it was inferred and marked as implicit (see the "cloud" in figure 2). The entire research team vetted each diagram, restructuring it until consensus was reached. A total of 277 episodes of argumentation were diagrammed, 177 from Kylee's class and 100 from Bridgett's class. After the structure of each argument was agreed upon, we followed the same process to insert the questions and other support that the teachers provided for the argument components. Finally, we inserted all of the information from the diagrams into spreadsheets, categorizing the types of support provided by the teachers and the kinds of explicit warrants provided in each class. We used an inductive analysis method in which the categories of support and the classifications of warrants emerged from the data, first using very specific codes and then merging those into larger categories. This paper examines the kinds of explicit warrants contributed by the teachers,

[1] All names are pseudonyms.

students, and jointly, in search of the kinds of reasoning that was supported and accepted in these classes.

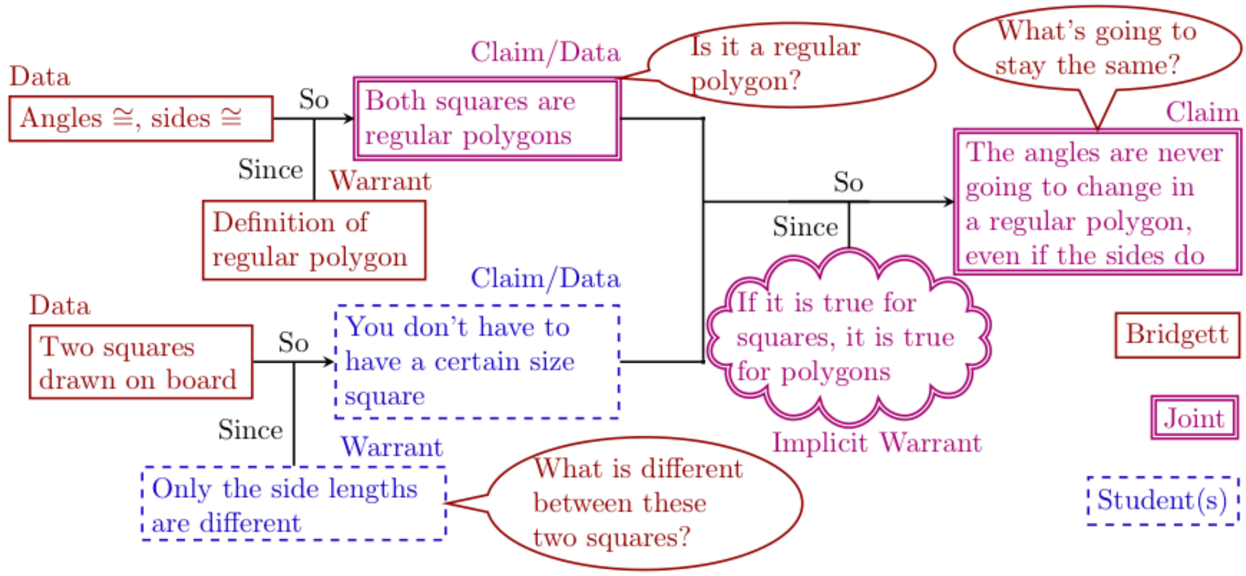

Figure 2: Example of argument from Bridgett's class

RESULTS

Analysis of the warrants and associated reasoning in these classes is ongoing. In our initial analysis, we found 29 different kinds of warrants, such as the appearance of a diagram, noticing a pattern, the statement or interpretation of a theorem, and calculations. By examining the coded data, we collapsed these into ten major categories: mathematical knowledge, verification, authority or external validation, interpretation, patterns, method, calculation, visual, unformalized mathematical knowledge, and given information. In general, fewer explicit warrants were contributed in Bridgett's class, and those that were contributed were generally contributed by students or by students and Bridgett together rather than by Bridgett by herself. In contrast, Kylee herself contributed over half of the explicit warrants in her class, with students alone or in combination with Kylee contributing about 47%.

We first examined all of the warrants aggregated across both classes. Of the 647 explicit warrants that were contributed, approximately 30% were classified as mathematical knowledge, a category that included definitions, theorems, and results that were previously established in the class. These warrants made up approximately 30% of the warrants in each of the teachers' classes (29% of Kylee's and 31% of Bridgett's) as well. In addition, approximately 27% of the explicit warrants were classified as calculations, which included such things as simple operations on numbers or solving straightforward equations. The proportion of such warrants was higher in Kylee's class (28%) than in Bridgett's (21%). The next highest category was visual (14%), a category that included the appearance of a figure, visual cues, such as markings on diagrams, and visualization, or mentally manipulating a diagram. This category was particularly prevalent in Kylee's class, with more than 14% of the explicit warrants in her class classified as visual; about 10% of the explicit warrants in Bridgett's class were classified as visual. All other categories accounted for the other 29% of the explicit warrants, but no other category alone made up more than 10% of the explicit warrants when data from both classes were combined. When Bridgett's class was examined separately from Kylee's, two other categories appeared to be important: unformalized mathematical knowledge (drawing on previous students' experiences with mathematics, number sense, intuitive understandings of mathematical ideas) accounted for 13% of the explicit warrants in her class and patterns appeared 10% of the time. These categories were much less prevalent in Kylee's classes.

When we examined the types of warrants within the mathematical knowledge category, we noticed that there were distinct differences between the types of warrants in each classroom. In Kylee's class, students and teacher alike referred to theorems and definitions very frequently. In fact, more than half of the warrants categorized as mathematical knowledge in her class were theorems and definitions. The theorems and definitions were formally presented, and Kylee herself contributed many of these. Kylee also frequently pressed her students to provide these, particularly when they were writing two-column proofs. On the other hand, Bridgett and her students were more likely to refer to results established in class that had not yet achieved the status of a theorem, such as the answer to a previous problem or a particular student's previous observation about a figure or pattern. The following two episodes illustrate the differences between the knowledge and its uses in each class.

Illustrative Example from Kylee's Class

Toward the end of class during one of our first observations of Kylee, she was illustrating the use of different triangle congruence theorems. In particular, she wanted to illustrate the "Angle-Angle-Side Congruence Theorem." So, as illustrated in Figure 3, Kylee drew a figure and some given information on the board and proceeded to develop a proof (which she wrote in two-column form) on the board with her students. One of the interesting things about this episode is that a student contributed a warrant ("Angle-Side-Angle") that was not appropriate given the stated congruent parts of the triangles, and Kylee replied, seemingly automatically, "Right, good." She then caught herself, remarking that "this is supposed to be angle-angle-side," and went on to establish the congruence by the intended theorem. This is captured in the diagram in Figure 3 with the student's warrant followed by the teacher's rebuttal and subsequent data and warrant leading to the final claim. In this episode, the warrants are primarily mathematical knowledge, and, in particular, properties and theorems. Students contribute two of the warrants. But, the preponderance of contributions were attributed to Kylee, and she appears to be the one who is doing the mathematical thinking. Her students contributed warrants, but when one of the warrants is not appropriate for the stated congruent parts, it is Kylee who re-examines the diagram and the congruent parts, and it is Kylee who states the correct theorem.

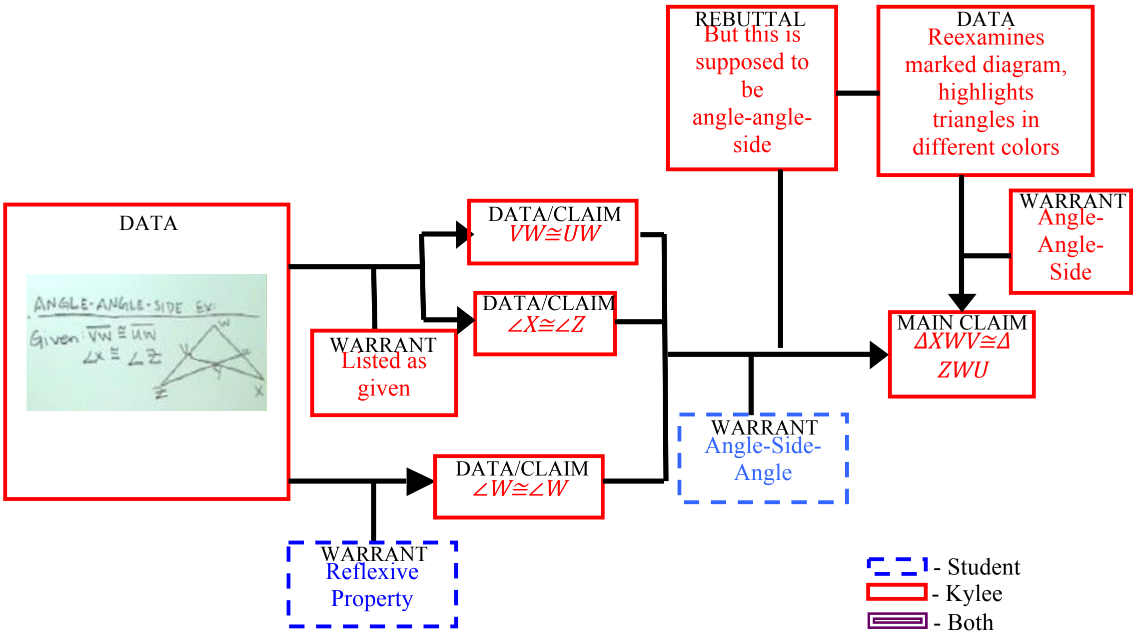

Figure 3: Argument from Kylee's Class

Illustrative Example from Bridgett's Class

As a contrast to the kinds of warrants observed in Kylee's class, Figure 4 depicts an argument that occurred in Bridgett's class. Bridgett's students had just completed an investigation in which they had examined the possible lengths for the sides of a triangle, and students had contributed two different versions of the triangle inequality theorem. Bridgett asked her students to determine whether three segments of lengths three, four, and seven could form a triangle. After a student said, "no," and gave data and the warrant, "the two smallest sides have to be as long as the longest side," Bridgett agreed and gave an additional data and warrant pair, citing the other version of the triangle inequality theorem and attributing it to a particular student.

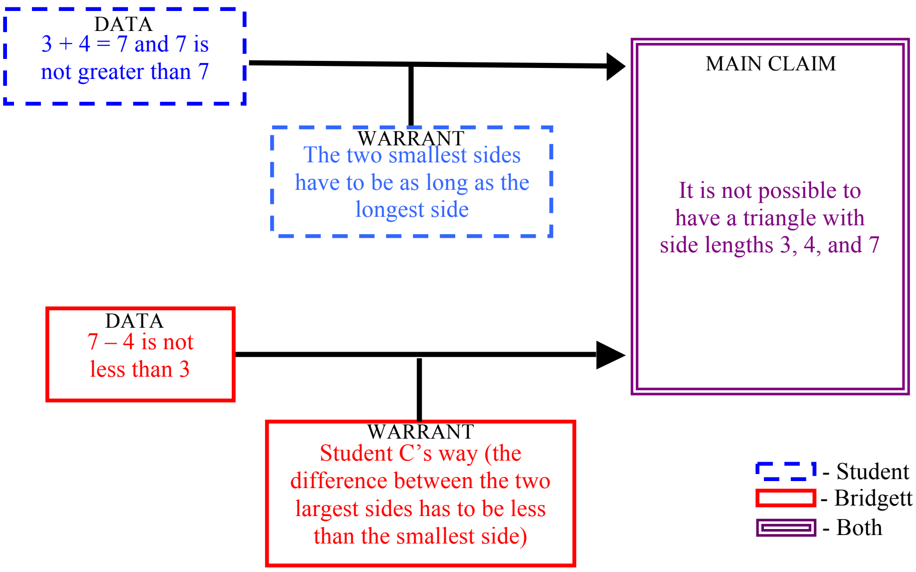

Figure 4: Argument from Bridgett's Class

Reasoning Patterns in the Two Classes

In both classes, established mathematical knowledge, similar to what Nardi, Biza, and Zachariades (2011) call *a priori-epistemological*, was accepted as appropriate justification for claims. However, in Bridgett's class, that mathematical knowledge was often established by the students and teacher together in previous arguments, while in Kylee's class, the mathematical knowledge used was often presented by Kylee as externally valid. Kylee and Bridgett employed different class structures and different teaching methods, with Kylee using primarily direct instruction and whole class discussions and Bridgett using primarily small group investigations followed by whole class discussions of the results of those investigations. In general, the warrants in Bridgett's class reflect a less formal pattern of reasoning, with more informal justifications for claims. These justifications were also more likely to be contributed by students, and appeared to be personally meaningful to them. The reasoning patterns in Kylee's class tended to be more formal, with Kylee providing or pressing students for specific theorems or definitions that had been introduced.

IMPLICATIONS

We characterized the reasoning patterns in the classes as more or less formal. As mathematics educators who are concerned with students progressing toward participating in proving mathematical statements, it would be easy to suggest that the more formal reasoning exhibited in Kylee's class is further along the continuum toward proof and that her teaching might be beneficial for students to learn to prove. However, Kylee contributed many of these warrants, and the theorems and definitions that were usually used as warrants were presented as established and unquestioned knowledge. Bridgett's students, with their less formal reasoning, still seemed to exhibit more intellectual autonomy as they used their intuitive understandings and results established in their class as justifications in their arguments. Even though when Bridgett engaged them in formally proving mathematical statements she had to scaffold their efforts, the less formal reasoning in Bridgett's class has more potential to prepare students for appropriately justifying mathematical ideas.

Yackel and Cobb (1996) suggest that one of the goals of mathematics education is to increase students' intellectual autonomy in mathematics. Examining the kinds of warrants accepted and supported in a classroom is one way to assess this goal as well as to describe the kinds of reasoning in which students and teachers are engaging. When we examine the sources of the mathematical knowledge that is used as warrants, we can differentiate between two classrooms in which the instruction seemed different even though the argumentation structures and warrant-types seemed similar on the surface.

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