Proposal for TSG14 in ICME2012

Introducing Proof in Lower Secondary School Geometry: a learning progression based on flow-chart proving

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Abstract

This paper reports on a learning progression based on flow-chart proving and aimed at providing a basis for introducing proof in lower secondary school geometry. The proposed learning progression has three phases: constructing flow-chart proofs in an open situation, constructing formal proofs by reference to flow-chart proofs in a closed situation, and refining formal proofs by placing them into flow-chart proof format in a closed situation. Trialling this approach in the classroom indicates that students who studied proof and proving with this learning progression might be better able to plan and construct a proof than other students who did not follow this approach.

1. Introduction

Even though the teaching and learning of proof is universally recognized as a key element of mathematics curricula, it remains the case that students at the lower secondary school level can experience difficulties in understanding proof (eg: Mariotti, 2006). Improving instructional approaches is one strategy and in our research we focus on the introductory phases of teaching proofs and proving.

Based on the idea of learning progressions as "successively more sophisticated ways of thinking about a topic that can follow one another as children learn about and investigate a topic" (NRC, 2007, p. 214), we designed a learning progression for the introductory phases of learning formal proofs in lower secondary school geometry. The purposes of this paper are: a) to provide a theoretical outline of some of the design

principles underpinning the introductory stage of proof in lower secondary school mathematics, and b) to evaluate the effects of implementing these lessons in terms of knowledge and understanding which students gained from the lessons. In terms of TSG14 theme, our paper is related to sub-topic 2: curriculum and textbook aspect (e.g. discussion of the mathematical contexts and developmental progression of RPP in curriculum) and sub-topic 3: cognitive aspect (e.g. description and interpretation of students' behaviors in RPP task).

2. Theoretical underpinnings

2.1. Conjecturing, refuting, and the nature of the proof construction process

Underpinning the design principles of our proposed learning progression are both the relationship between proofs and refutations (Lakatos, 1976) and the nature of the proof construction process (McCrone and Martin, 2009).

2.2 Developing learning progressions

As Empson (2011, p574) explains "the idea of learning progressions .. is now virtually synonymous with learning trajectory" Given our goal of researching the introductory learning of formal proof in lower secondary school geometry, we borrow from the notion of 'hypothetical learning trajectory' (HLT) that it includes "the learning goal, the learning activities, and the thinking and learning in which the students might engage" (Simon, 1995, p. 133). This set of components can underpin the design of a sequence of teaching; see, for example, Clements and Sarama (2004), Simon and Tzur (2004), Stylianides and Stylianides (2009). In terms of the introductory learning of formal proof in lower secondary school geometry, our learning progression comprises the following components:

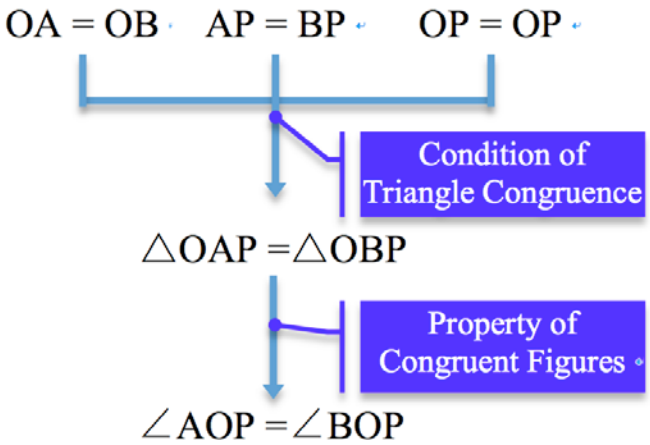
- Learning goals: by the end of the teaching, students will be able to a) plan and construct a proof in geometry, and b) understand the structure of proofs in geometry; this entails students beginning to grasp elements of the structure of proof, and then gradually being able to see the entire structure (Miyazaki and Fujita, 2010);
- The learning process and activities: proof construction based on the

flow-chart proof format (McMurray,1978), with both open and closed problem situations;

- The thinking and learning in which the students might engage: this encompasses analytic and synthetic thinking, planning and constructing a proof, reflection of the structure of proofs, and so on.

2.3 Flow-chart proving in an open problem situation

A key feature of our lessons is the use of flow-chart proofs that provides a visual representation of the deductive chain between premises and conclusion and the relations between singular and universal propositions. As McMurray (1978) and others have suggested, flow-chart proving can be introduced to students before they learn the 'two column proof' format.



A flow-chart proof shows a 'story line' of the proof; beginning with the kinds of assumptions from which the conclusion is deduced, and including the kinds of theorems being used, how the assumptions and conclusion are connected, and so on. To date our research findings suggest that flow-chart proofs can enable students to understand the structure of proof (Miyazaki & Fujita, 2010).

We consider that the power of flow-chart proofs can be particularly enriched in 'open' situations where students can construct multiple solutions by deciding the assumptions and intermediate propositions necessary to deduce a given conclusion. While this open situation should encourage students to think analytically, of course some may struggle to think synthetically based on the assumptions. Finally, students are expected to connect these two directions of thinking to plan a proof.

3. Developing a learning progression for introductory proving

Our learning progression for the introductory learning of formal proofs by using flow-chart proof in Grade 8 (aged 14) has three steps: constructing flow-chart proofs in an open situation, constructing formal proofs by reference to flow-chart proofs in a closed situation, refining formal proofs by placing them into flow-chart proof format in a closed situation. We explain the reasons for these steps below.

In the first phase, students construct flow-chart proofs in open situations. Through their activity they are expected to learn how to think analytically and synthetically *and* to understand the structure of proof. In the second phase, students construct formal proofs by reference of flow-chart proofs that they have already made in close situations. They then transform their flow-chart proof into a 'paragraph proof'. Here, students are expected to learn how to express the "arrows" of flow-chart proof by using mathematical language.

Finally, in the third phase, students refine formal proofs by placing them into flow-chart proof format in close situations. The reason for this third phase is that although students get familiar with constructing formal proofs gradually until the end of second phase, they have yet to have the opportunity of constructing them without flow-chart proofs. Therefore, they usually make some mistakes in their formal proofs. By channelling their formal proofs into flow-chart proof format, students have a chance to find their mistakes related to the structure of proof, and they will be in the habit of looking back over proofs and making them better.

By using our proposed learning progression of flow-chart proving, we designed nine lessons in cooperation with classroom teachers, considering the situation of proving, the steps of deductive reasoning, and the complexity of problem situation and proof as follows:

Learning Progression	No.	Situation of proving	Steps of reasoning	Characteristics of problem
Constructing flow-chart proofs	1	Open	1	Constructing flow-chart proofs by using congruency to two triangles connected by a vertex
	2	Open	1	Constructing flow-chart proofs by using congruency to two triangles connected by a side
	3	Open	1	Constructing flow-chart proofs using congruency to two triangles connected by a vertex with clarifying figural properties as assumptions
	4	Open	2	Constructing flow-chart proofs using congruency to two triangles connected by a side with clarifying figural properties as assumptions
Constructing formal proofs by reference of flow-chart proofs	5	Close	1	Constructing formal proofs with flow-chart proofs in the problem of Lesson No.3
	6	Close	2	Constructing formal proofs with flow-chart proofs in the problem of Lesson No.4
Refining formal proofs by replacing them into flow-chart proof format	7	Close	2	Constructing formal proofs and refine them with flow-chart proofs in the problem of Lesson No.6
	8	Close	2	Constructing formal proofs by using congruency to two triangles overlapped each other and refine them with flow-chart proofs
	9	Close	2	Constructing formal proofs by using properties of parallel lines and congruency to two triangles connected by a vertex and refine them with flow-chart proofs

4. Evaluating the learning progression for introductory proving

During 2011 three mathematics teachers in a public junior high school in Japan implemented the set of lessons based on our learning progression. Each teacher taught one class. In total, 94 students were taught. We collected both quantitative and qualitative data. While more details can be presented in the full version of this paper, in the space we have now we provide selected quantitative data that illustrate how our proposed learning

progression can help establish students' understanding of proof at the early stages (for qualitative data that students in Grade 8 can engage rich mathematical discussions with Flow-chart proofs, see Miyazaki and Fujita, 2010).

The quantitative data come from a survey we conducted with the students who learnt geometrical proofs with our learning progression. We used the same form of test items as the National Assessment taken by all students in Japan. To ensure the quality of the marking, the marking of our survey was conducted by the same organization that marked the National Assessment.

We chose five basic geometrical problems and one advanced problems related to geometrical formal proof. The full details of our survey results are provided in our full paper. Here we show one of the advanced problems that aims to check whether students can construct a formal proof based on a suggested plan of proving.

Problem B4

Takuya is trying to solve the following question.

Question

> In fig. 1, let us take points A, B, C and D on ∠XOY so that OA=OB and OC=OD.
> When A and D, and B and C are connected, prove AD=BC.

Takuya described his plan to prove it as follows.

Takuya's memo

> #1 To prove AD=BC, I have to show △AOD≡△BOC.
> #2 To see △AOD and △BOC more clearly, I can divide the fig. 1 into two parts and identify what are assumed.
> #3 By using these ideas, I think I can prove △AOD≡△BOC.

(1) As we can see #1 in Takuya's memo, to prove AD=BC, we have to show △AOD≡△BOC, and which property should he use to complete his proof? Choose from a)-d).

a) In congruent figures, corresponding sides are equal.
b) In congruent figures, corresponding angles are equal.
c) In congruent figures, perimeters are equal.
d) In congruent figures, areas are equal.

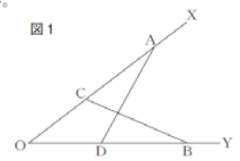
(2) Prove AD＝BC in the problem on the left page

4 拓也さんは，次の問題を考えています。

問題

下の図１のように，∠XOY の辺OX と辺OY 上に，OA＝OB となるように点Aと点Bを，OC＝ODとなるように点Cと点Dを，それぞれとります。
点Aと点D，点Bと点Cをそれぞれ結ぶとき，AD＝BCとなることを証明しなさい。

図１

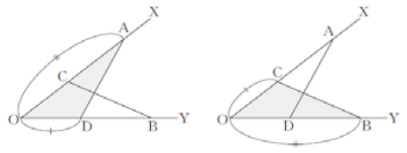


拓也さんは，証明の方針を下のようなメモにまとめました。

拓也さんのメモ

① AD＝BC を証明するためには，△AOD と△BOC の合同を示せばよい。

② 図１の△AOD と△BOC を見やすくするために，２つの図に分けて，仮定を表すと，下のようになる。



③ ②をもとにすると，△AOD と△BOC の合同が示せそうだ。

（1）拓也さんのメモの①にあるように，AD＝BC を証明するために，△AOD と△BOC の合同を示せばよいのは，合同な図形のどのような性質からですか。下のアからエの中から１つ選びなさい。

ア 合同な図形の対応する辺の長さは等しい。
イ 合同な図形の対応する角の大きさは等しい。
ウ 合同な図形の周の長さは等しい。
エ 合同な図形の面積は等しい。

（2）前ページの問題で，AD＝BC となることを証明しなさい。

The below table summarizes the correct answers (%) of our samples and the National average for the two questions given above.

	Q (1)	Q (2)
National average	63.3	43.3
Our sample	73.3	48.9

In terms of the issue of proof construction, the above data show that 73.3% and 48.9% of the students in our study answered correctly; this is 10 and 5.6 points higher than the national average. This indicates that our students who studied proof and proving with our learning progression might be able to plan and construct a proof better than other students. The full details of our quantitative data will be presented in our full version of paper and presentation. As a tentative conclusion, we consider that the teaching of proof and proving with flow-chart proof based on our learning progression will work effectively as an introductory instruction. In addition, we also have qualitative data on how our students learnt flow-chart and formal proofs within our learning progression (using trajectory terminology, this is their 'Actual Learning Trajectory'; see, Simons, 1995).

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