# How Prospective Mathematics Teachers Decide if an Argument is a Valid Proof: Perception and Practice

**Dr. Nermin Bayazit, Georgia State University, USA**

**Abstract**

In this presentation, I will report a study done to explore prospective secondary mathematics teachers' perception of what constitutes a proof and how they assess the validity of a proof, and to investigate the relation, if there is any, between their perception and practice. An hour to 90 minutes interviews were conducted with six senior prospective secondary mathematics majors for data collection. The results revealed that the participants' perception of proof is aligned with mathematicians that proof should explain why something works. Also, none of the participants accepted empirical arguments as a valid proof regardless of the number of examples used. Lastly, for all participants, the criterion for a valid proof was logical deduction.

## Introduction

Research has demonstrated that there is a consensus on the integral role of proof in mathematics among mathematicians and mathematics educators (Hanna, 2000; Jones, 1997; NCTM, 2000; Schoenfeld, 1994). On the other hand, there are a considerable number of research studies showing that students do not see proof as an essential part of mathematics (e.g. Alibert, 1988; Coe & Ruthven, 1994; Knuth, 2002; Schoenfeld, 1995). Unfortunately, most students see proof as a redundant activity which is done to confirm already known facts (Coe & Ruthven, 1994). This perception is considered to be critical in terms of their difficulties in doing proof (Furingetti, Olivero & Paola, 2001).

Reform-based mathematics instruction places significant emphasis on having students develop and evaluate mathematical arguments and proofs (NCTM, 2000). On the other hand, educational research has shown that both students and teachers have difficulties in these practices. Morris (2007), for example, found that prospective elementary teachers rarely use logical validity as a criterion for evaluating arguments. In a similar vein, Knuth (2002) found that in-service high school teachers would accept flawed arguments as proofs if they were in an appropriate format. Selden and Selden (2003) and Alcock and Weber (2005) found that mathematics majors, contrary to their intentions, performed at chance level when asked to determine if a proof is valid. Lastly, Weber (2009) stated that unlike prospective elementary teachers, mathematics majors who took a proof transition course do not accept empirical arguments as valid proofs. Although proof perception and validation practices are well studied, there has not been research focusing on special group, namely prospective secondary mathematics teachers. I believe that it is critical to study this community who are the students of today and teachers of tomorrow who will interact most with students during their proof experiences, often in geometry context. With this in mind, the study reported here was done to explore prospective secondary mathematics teachers' perception of what constitutes a proof and how they assess the validity of a proof. The goals of the validity assessment practice were to identify the participants' criteria for a valid proof and to investigate the relation, if there is any, between their perception and practice.

## Methodology

**Participants:** The study was conducted with six senior secondary mathematics education majors from a large southeastern university in the U.S. All of the participants had Calculus I, II, and III, Linear Algebra, Geometry and a transition to proof course at undergraduate level. Four of the participants were traditional students who attended college right after their high school education. Other two participants were career charger.

**Data Collection and Analysis:** Data for this study was collected qualitatively through semi-structured interviews. The researcher met with the participants individually to conduct the interviews which lasted 1-1.5 hours. Students were asked several questions to understand their perceptions about mathematics, proof, mathematical definitions, role of proof in mathematics, validity of a proof. They were also given different student arguments for two propositions to assess the validity of the arguments. Member check and peer-evaluation was used for the validity of the interpretations.

## Results

### What is a proof?

When the participants were asked about their understanding of proof, the common perspective was that it is the way to show why “something” works. All students emphasized the role of proof as to “explain why”. Proof has been perceived as the answer for all possible questions about the statement. Following are some quotes in this regard:

Chris: It is basic. It covers all possible side arguments. You can explain and answer all the related arguments.

Laura: When you are trying to argue a point and if you say this is the way it should be somebody will ask you “why?” and you have to be able to break down your argument and say “well, this is why?”. You need to be able to show why it is true.

This result is not consistent with previous researches which showed that a majority of students see the role of proof as to verify an algorithm or a statement ( Healy and Hoyles, 2000; Varghese, 2009). In this study, the group’s perspective was more aligned with of mathematicians. They were interested in “more than whether a conjecture is correct, they want to know why it is correct” (Hersh, 1993). Three of the students stated proof as a way of showing their understanding of the concept. Adam, for example argued that
“proof is a great way to teach mathematics. Because if you do proof with your students, walk through with them, they can see why that works. They can maybe better understand the concepts rather than just knowing the concept or idea. You can know something and you can understand something”. He further stated that one of the roles of proof is to discover mathematics which is not a common view among students. Only one student mentioned validation as a role of proof besides explaining. He argued that proof is done to validate already known facts and explain why they work.

### Structure of a proof

All of the participants described their proof approach as starting with the givens and “working” with them to reach the conclusion. They claimed to use all the givens to have a “successful proof”. In the middle stage, they had different approaches to find the way to the conclusion. One student, Chris, stated that he looks for the key words to identify the best proof method to employ. He explained that when a statement uses universal quantifier, he first tries induction method. However, if the statement is in the form of “if…then” he tends to use direct proof first, and if it does not work then contradiction or contrapositive. On the other hand, Brad argued to manipulate the givens to reach the conclusion. He stated that “a lot of my proofs are just because of givens…you have to see what you are looking at if it is something that you can manipulate most likely it is going to be some kind of direct proof because all you have to do is to figure out a way to manipulate it to get it look line what you want it to look like”. His approach was consistent with his perception of proof as validating the already known facts with explaining each step.

**Validity of a proof:**

For all participants, the criterion for a valid proof was logical deduction. They focused on whether each statement in the proof followed from previously established truths or the assumptions of the proposition through logical reasoning. They agreed that a proof can be invalid due to the unfounded assumptions or incorrect line of reasoning. Their practice of evaluating the validity of a given proof was aligned with their perception of a valid proof. Discussions on the validity of the student arguments provided in Appendix A and Appendix B exemplified the consistency. For instance, none of the participants accepted the Student Argument #1 for proposition 2 as a valid proof. Although, it was presented in the most familiar structure, two-column proof, they criticized the flow of the statements as not being logical. Moreover, they analyzed the explanation of each statement and checked for its validity. For example, they all stated that starting from step five the provided reasons do not support the statements. Some of them put check marks next to each reason if it made sense or not. Four of them made the argument that the drawing was not scaled so nobody can make assumptions about it. Emily, for instance, commented that “I am so bad with assuming and do it all the time and Dr. Johnson (pseudonym) always warns me about it”.

Unlike many research findings (e.g. Martin&Harel, 1989; Morris, 2007) none of the participants accepted empirical arguments (#1 and 4 for proposition #1 see Appendix A) as a valid proof. They all argued that both of them only provide specific examples that verify the "proposition works" but it does not guarantee that it will always be true. They also made the clarification that student argument #4 is logically the same with argument #1only with more examples. However, four of the participants stressed the importance of example based reasoning for personal conviction. Lily, for example, stated that "this is the proof, this is the explanation. Here are some numbers that you can try, but I do not think it would be a proof by itself. But they can help them arrive at the proof." Weber (2009) also found a similar result with mathematics majors. In his research, 26 out of 28 participants also did not find empirical arguments as convincing. He hypothesized that an explanation for this result would be the participants' experience in a transition to proof.

Participants of the study tend to refuse filling the gaps of a "dense proof" (Solow, 2000). Although they could make the connection between two steps, they criticized students not providing them in the proof as seen in student argument#2 for Proposition 2 (See Appendix B).

When they were asked the question "who would be the person to ask to check the validity of their proof", only Emily had the limited perspective of having her teachers as the final authority. For the other five participants, although all of them had the idea that their mathematics instructors could be one person to ask to check the validity of their proof, the bottom line was having the "appropriate knowledge in and experience with mathematics" or being "higher mathematical thinkers".

Lastly, although all of the participants had serious hesitation about their ability to do mathematical proofs, they all developed a strong perception of how a valid proof should look like. The results of the analysis of their perception of a valid proof are encouraging as being parallel to what mathematicians have. As it was stated by most of the participants, their lack of experience with mathematical proof negatively affects their confidence in providing, even working on constructing formal proofs.

## Conclusion

The results of this study is encouraging in the sense that at least this group of prospective secondary mathematics teachers' perception about proof is aligned with mathematicians that proof is important to understand why something works. They all value the role of proof in mathematics and acknowledge its explanatory role. Moreover, none of the participants accepted empirical arguments as a valid proof. Despite the encouraging results, we still face the problems of not liking proof "since it is hard", having limited experience with actually doing proofs and discussing the validity of proofs, and the "if and only if" relation between the insecurity in the ability of constructing proofs and being inexperienced in doing it. One way to address this issue for secondary mathematics education curriculum would be to include a "transition to proof" course early in their programs so that students would have the necessary tools to work with proof in different contexts through their degree. As they have more experience with proof they might be more confident in their ability to do them and eventually integrate more in their teaching when they are in the field. Also, proof validation activities offer a unique space for them to be exposed to several different proofs without actually constructing them. As they have more of these activities, they would have more examples to refer to, use or even mimic when they try to prove a statement.

## References


Alcock, L. & Weber, K. (2005). Proof validation in real analysis: Inferring and checking warrants. *Journal of Mathematical Behavior,* 24, 125-134.

Alibert, D. (1988). Towards new customs in the classroom. *For the Learning of Mathematics, 8*(2), 31-35.

Coe, R., & Ruthven, K. (1994). Proof practices and constructs of advanced mathematics students. *British Educational Research Journal, 20*(1), 41-53.

Furinghetti, F., Olivero, F., & Paola, D. (2001). Students approaching proof through conjectures: Snapshots in a classroom. *International Journal of Mathematical Education in Science and Technology, 32*(3), 319-335.

Hanna, G. (2000). Proof, explanation, and exploration: An overview. *Educational Studies in Mathematics, 44,* 5-23

Healy, L., & Hoyles, C. (2000). A study of proof conceptions in algebra. *Journal for Research in Mathematics Education, 31*(4), 396-428.

Hersh, R. (1993). Proving is convincing and explaining. *Educational Studies in Mathematics, 24,* 389-399.

Jones, K. (1997). Student teachers' conceptions of mathematical proof. *Mathematics Education Review, 9,* 21-32.

Knuth, E. J. (2002). Teachers' conceptions of proof in the context of secondary school mathematics. *Journal of Mathematics Teacher Education, 5,* 61-88.

Martin, W.G., & Harel, G. (1989). Proof frames of pre-service elementary teachers. *Journal for Research in Mathematics Education, 20,* 41-51.

Morris, A. K. (2007). Factors affecting pre-service teachers' evaluations of the validity of students' mathematical arguments in classroom contexts. *Cognition and Instruction, 25*(4), 479-522.

National Council of Teachers of Mathematics (2000). *Principles and Standards for School Mathematics.* Reston, VA: Author.

Schoenfeld, A. H. (1994). What do we know about mathematics curricula? *Journal of Mathematical Behavior, 13*(1), 55-80.

Schoenfeld, A. (1995). *Mathematical Thinking and Problem Solving*. Lawrenceville, NJ: Erlbaum.

Selden, A., & Selden, J. (2003). *Errors and Misconceptions in College Level Theorem Proving* (Tech. Rep. No. 3). Cookeville, TN: Tennessee Technological University, Mathematics Department.

Weber, K. (2009). How syntactic reasoners can develop understanding, evaluate conjectures, and construct counterexamples in advanced mathematics. *Journal of Mathematical Behavior*, 28, 200-208

Varghese, T. (2009). Secondary level student teachers' conceptions of mathematical proof. *Issues in the Undergraduate Mathematics Preparation of School Teachers*, 1, 1-13.

## Appendix A

Proposition 1: Prove that the sum of 3 consecutive integers is divisible by 3

**Student argument # 1**

$-1 + -2 + -3$
$-6$
$\frac{-6}{3} = -2$ ✓

$7 + 8 + 9$
$24$
$\frac{24}{3} = 8$ ✓

**Student argument # 2**

let n be any integer
$n-1, n, n+1 = 3n$ so there's a factor of 3, therefore it's divisible by 3.

**Student argument # 3**

Examples: $1+2+3=6$ $6/3=2$ and $11+12+13=36$ $36/3=12$
By adding 3 consecutive integers, it is the same as multiplying the second integer by three.
Ex $1+2+3$ (subtract 1 from 3) now you have $2+2+2$ (witch is the same as dividing by 3.

**Student argument # 4**

$1+2+3=6$ $6/3=2$
$4+5+6=15$ $15/3=5$
$6+7+8=21$ $21/3=7$
$10+11+12=33$ $33/3=11$
$16+17+18=51$ $51/3=17$
$19+20+21=60$ $60/3=20$
$22+23+24=69$ $69/3=23$
$30+31+32=93$ $93/3=31$
$34+35+36=105$ $105/3=35$

**Student argument # 5**

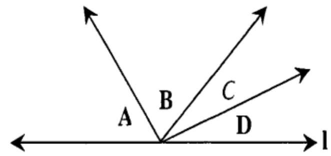
$\frac{x+(x+1)+(x+2)}{3} = 0$
$2x+1 + x+2 = 0$
$3x+3=0$

$3x=$
$3(x+1)=0$

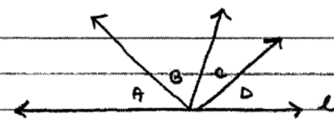
Figure 1: Student arguments for the proof of Proposition 1

## Appendix B

Proposition 2: Given that $l$ is a line, $\angle A \cong \angle B$ and $\angle C \cong \angle D$, prove that $m\angle A + m\angle C = 90°$ .




**Student argument # 1**




| Statements | Reason |
|---|---|
| 1.) $l$ is a line | 1.) given |
| 2.) $\angle A \cong \angle B$ | 2.) given |
| 3.) $\angle C \cong \angle D$ | 3. given |
| 4.) $m\angle A + m\angle B + m\angle C + m\angle D = 180°$ | 4.) angle addition property |
| 5.) $m\angle A + m\angle B =$ obtuse $\angle$ | 5.) def. of obtuse $\angle$ |
| 6.) $m\angle C + m\angle D =$ acute $\angle$ | 6.) def. of acute $\angle$ |
| 7.) $\angle B =$ midpoint of line $l$ | 7.) def. of midpoint |
| 8.) $m\angle B + m\angle D = 90°$ | 8.) def. of right $\angle$ |
| 9.) $m\angle A + m\angle C = 90°$ | 9.) " " |

**Student argument # 2**

$m\angle A + m\angle C + m\angle B + m\angle D = 180°$

$m\angle A + m\angle A + m\angle C + m\angle C = 180°$

$m\angle A + m\angle C = 90°$

**Student argument # 3**

If any two $\angle$'s are $\cong$ w/ each other and the other two are as well. The line = 180° then any two $\angle$'s added together = 90° because of a Thereom that was stated in Geometry.

Figure 2: Student arguments for the proof of proposition 2