Post-ICME monograph:
Advances in mathematics education research on proof and proving: An international perspective

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Theme 1: Epistemological issues related to proof and proving

Reflections on proof as explanation, **G. Hanna**

This paper explores the connection between two distinct ways of defining mathematical explanation and thus of identifying explanatory proofs. The first is the one discussed in the philosophy of mathematics, in which a proof is considered explanatory when it helps account for a mathematical fact, clarifying why it follows from others. It is concerned with intra-mathematical factors, not with pedagogical considerations. The second definition is the one current among mathematics educators, who consider a proof to be explanatory when it helps convey mathematical insights to an audience in a manner that is pedagogically appropriate. This latter view brings cognitive factors very much into play. The two views of explanation are quite different. The paper shows, however, citing examples, that insights from what are considered by philosophers of mathematics to be explanatory proofs can sometimes form a basis for explanatory proofs in the pedagogical sense and thus add value to the curriculum.

Working on proofs as contributing to conceptualization - The case of IR completeness-prolegomena to a didactical study, **V. Durand-Guerrier, D Tanguay**

In this communication, we propose a mathematical and epistemological study about two classical constructions of the real numbers system, and the associated proofs of its completeness. In addition, we present two contrasting constructions relying on decimal expansions. The general didactical issue pertains to the potential contribution of analyzing proofs as a means for deepening the understanding of the objects in play. This study falls within a larger project about the conceptualization of real numbers, taking into account the triad discreteness/density/continuity.

Types of epistemological justifications, **G. Harel**

We distinguish among three types of epistemological justification:

1. *(1) Sentential epistemological justification (SEJ). This refers to a situation when one is aware of how a definition, axiom, or proposition was born out of a need to resolve a problematic situation.*
2. *(2) Apodictic epistemological justification (AEJ). This pertains to the process of proving. It is when one views a particular logical implication, $a \rightarrow b$, in causality, or explanatory,*

terms—how a causes b to happen. This can take place in two forms. One might observe a, asks what are its possible consequences, and finds out that b is a consequence of it. Or one might observe b, asks about its causes, and finds out that a is a cause of it.

(3) Meta epistemological justification (MEJ). This refers to a situation when one not only possesses SEJ and AEJ, but also he or she is aware of how the sentence or the implication came into being.

These three types will be illustrated with examples from the field of complex numbers

Reasoning and proof in elementary teacher education: The key role of the cultural analysis of the content, **P. Boero, G. Fenaroli, E. Guala**

The problem dealt with in this paper concerns how to prepare prospective elementary teachers to develop students' argumentative skills in school, in spite of difficulties deriving from present school culture and past teacher education in Italy. The salient features of a course on mathematical argumentation, aimed at making prospective elementary teachers free from those influences and able to perform autonomous professional choices, are described. The development of the competence of Cultural Analysis of the Content to be taught (CAC) is assumed and motivated as a condition for teachers' professional autonomy. Specific educational choices and some results concerning the development of participants' CAC in the course at stake are presented and discussed.

Theme 2: Classroom-based issues related to proof and proving

Constructing and validating a mathematical model: The teacher's prompt, **M. Goizueta, M. A. Mariotti**

Drawing on the hypothesis that an epistemology of school mathematics is interactively constituted in the classroom, we assume that different epistemological stances may lead students to get differently involved in the production and evaluation of arguments as part of their mathematical activity. In this paper we focus on how students exploit teacher's interventions to produce arguments to validate different mathematical models within a problem-solving situation. We show that teacher's interventions do not have the intended effect, in spite of their potential to foster students' reflection upon the adequacy of these models to the proposed empirical situation. Instead, a particular interpretation of the situation emerges as a consequence of a model *ultimately validated by the teacher; a phenomenon we call ex post facto modeling. We depart from this phenomenon to discuss some aspects of the mathematical culture of the classroom.*

Classroom-based interventions in the area of proof: Addressing key and persistent problems of *students' learning*, ***A. J. Stylianides, G. J. Stylianides***

While research has provided a strong empirical and theoretical basis about major difficulties students face with proof, it has paid less attention to the design of interventions to address these difficulties. We discuss the need for more research on classroom-based interventions in the area of proof, and we raise the question of what might be important characteristics of interventions that specifically aim to address key and persistent problems of students' learning in this area. In particular, we draw on prior research to make a case for interventions with the following three characteristics: (1) they include an explanatory theoretical framework about how they "work" or "can work" in relation to their impact on students' learning; (2) they have a narrow and well-defined scope, which makes it possible for them to have a relatively short duration; and (3) they include an appropriate mechanism to trigger and support conceptual change in students.

How can a teacher support students in constructing a proof? **B. Pedemonte**

This chapter analyzes the one-to-one interaction between student and teacher when student is engaged in constructing a geometrical proof. This analysis shows that it is not easy for the teacher to modify the student's argumentation based on conceptions that can hardly evolve into theorems. The teacher's intervention can be considered effective if it doesn't completely "interrupt" cognitive unity between the student's argumentation and proof, but opposite it encourages the continuity between them. Toulmin's model, used to analyze the student's argumentation and the teacher's intervention, highlights that the teacher's intervention needs to become a rebuttal in student's argument to invalidate it. The incorrect argument is refused by student only if the teacher's rebuttal has the same backing of the student argument and it is "coherent" with the student warrant.

Proof validation and modification by example generation: A classroom-based intervention in secondary school geometry, **K. Komatsu, T. Ishikawa, A. Narazaki**

Recent curriculum reforms underline mathematical activity related to proof validation, but few studies have explicitly addressed proof validation at the secondary school level. This paper *reports on our study of this issue. We suggest a specific kind of task for introducing proof validation in secondary school geometry and define the meanings of proof validation and proof modification in terms of Lakatos' notion of the local counterexample. We briefly report on a classroom-based intervention implemented using such tasks in a lower secondary school in Japan. We then analyse the results of a questionnaire conducted after the intervention to investigate how well the students did in proof validation and modification. The analysis shows that student failure in proof validation arose mainly from their difficulty with producing diagrams that satisfied the condition of the proof problem.*

Theme 3: Cognitive and curricular issues related to proof and proving

Mathematical argumentation in pupils' written dialogues, **G. A. Askevold, S. Lekaus**

In this article we present some results from a project about mathematical argumentation and proving in form of dialogues. Tasks are prepared in the form of written dialogues between imaginary pupils discussing mathematical problems, and pupils are invited to write their own dialogues continuing the mathematical discussion. We analyse dialogues about fractions written by pupils from two classrooms in Norway in grades 5 and 6. We show that many of the 5th grade pupils were strongly committed to visual representations of fractions in their argumentation, while the 6th graders used more rule-bound approaches. The analysis will show that the pupils use both diagrammatic and narrative argumentation. We will relate this to the theory of relational and instrumental understanding in mathematics.

Allowance by experts for a break in "linearity" of deductive logic in the process of proving, **S. S. Karunakaran**

Mathematicians have long claimed that the proving process cannot be considered a "linear" process and that undergraduates may view the proving process to be necessarily "linear". However, there is little empirical research that supports this familiar claim. Using grounded theory methods, "expert" provers of mathematics were examined in the process of proving novel mathematical statements. Expert provers of mathematics were willing to knowingly and temporarily interrupt the deductive logic of their proving process in order to make progress towards constructing an eventually complete deductive argument.

Reasoning-and-proving in school mathematics textbooks: A case study from Hong Kong, **K.-C. Wong, R. Sutherland**

To promote learning mathematics with understanding, mathematics educators in many countries recommend that proof (and proof-related reasoning) play a central role in school mathematics. In response to this recommendation, this study examines the opportunities for students to learn reasoning-and-proving from solving algebra problems in a popular school mathematics textbook from Hong Kong. The study adopts the methodology of Stylianides (2009). Results show that such opportunities are relatively limited. This suggests that proof plays a marginal role in school mathematics in Hong Kong

Irish Teachers' Perceptions of Reasoning-and-Proving Amidst a National Educational Reform, J. **D. Davis**

The syllabi driving the secondary mathematics education reform in Ireland expect students to engage in two components of reasoning-and-proving (RP) (Stylianides, 2008): making mathematical generalizations (pattern identification and conjecturing) and providing support to mathematical claims (providing a proof/non-proof argument). This study examines the perceptions of pattern identification, conjecturing, and proof by 22 Irish teachers with varying levels of teaching experience via semi-structured interviews. These teachers perceived pattern identification and conjecturing as disconnected from proof construction. Indeed, teachers struggled to define conjecturing and proof. There also appeared to be a bifurcation in students' classroom experiences with RP processes. Teachers stated that students with perceived lower ability levels experiences with proof ended at pattern identification while higher-level students rarely engaged in pattern identification and focused on memorizing proofs due to the influence of high stakes assessments. The implications of these results are discussed.

Theme 4: Issues related to the use of examples in proof and proving

How do pre-service teachers rate the conviction, verification and explanatory power of different kinds of proofs? **L. Kempen**

In the opening session of a course for first-year secondary (non grammar schools) pre-service teachers, the participants were asked to rate the conviction, verification and explanatory power of four different kinds of proofs (a generic proof with numbers, a generic proof in the context of figurate numbers, a proof in the context of figurate numbers using "geometric variables", and the so-called formal proof). The results may open the discussion about the concept of proofs that explain with regard to the explanatory power of the mathematical symbolic language.

When is a generic argument a proof? **D. Reid, E. V. Vargas**

We consider whether a generic argument can be considered a proof. Two positions on this question have recently been published which focus on the fussiness of an argument as a deciding criterion. We take a third view that takes into account psychological and social factors. Psychologically, for a generic argument to be a proof it must result in a convincing deductive reasoning process occurring in the mind of the reader. Socially, for a generic argument to be a proof it must conform to the social conventions of the context. For classroom settings, we suggest two kinds of evidence that should be reflected in written work in order for a generic argument to be accepted as a proof. These kinds of evidence reveal the linkage between the psychological and social factors.

Systematic exploration of examples as proof: Analysis from four theoretical perspectives, **O. Buchbinder**

This paper offers a multi-layered analysis of one specific category of students' example-based reasoning, which has received little attention in research literature so far: systematic exploration of examples. It involves dividing a conjecture's domain into disjoint sub-domains and testing a single example in each sub-domain. I apply four theoretical perspectives to analyze student data as a way to deepen and broaden insights gained from the analysis of this phenomenon. Implications for teaching and learning of proof in school mathematics are discussed.

Use of examples of unsuccessful arguments to facilitate students' reflection on their proving processes, **Y. Tsujiyama, K Yui**

Proving is an essential component in mathematical activities, but a difficult one for many students. One reason for this might be that unsuccessful arguments during the process of

planning a proof do not appear in a completed proof, and students cannot see how those arguments influenced the proof. If students can reflect on such arguments, they would be able to learn about proving, and effective ways to derive a proof. In previous studies, worked examples that show successful processes of deriving a proof are provided to deepen students' understanding of proving. However, such examples do not include unsuccessful arguments. This study examines how examples of unsuccessful arguments can facilitate students' reflection on their process of planning a proof by designing, implementing, and analysing an eighth-grade geometry lesson. It was found that an example of unsuccessful arguments enabled the student to comprehend the reason why the unsuccessful arguments failed, and why the successful ones worked.