THE VIEW OF MATHEMATICS AND ARGUMENTATION

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The view of mathematics effects on how the role and function of argumentation is seen. If mathematics is considered as an axiomatic system, arguments have to be based on definitions, axioms and previously proven theorems, and the function of argumentation is mainly to verify and systematise a statement. On the other hand, if mathematics is seen as a thinking and learning process, which requires conceptual and holistic understanding, arguments based on more concrete representations are important. In that case, the function of argumentation is to engender understanding by explaining. These two types of argumentation provide a framework, which could be applied in the design of teaching.

Key words: argumentation, formal and informal, mathematical views, representation.

INTRODUCTION

What kind of arguments should be favoured and appreciated in mathematics education? What kinds of arguments should students learn to understand, produce and present? These are important questions in design of teaching practices, teaching materials, assessment etc. In this paper some aspects to these questions will be presented on the basis of different views of mathematics and some earlier studies about mathematical reasoning.

Arguments can be seen both as elements and as products of a mathematical reasoning process. Often an aim of a reasoning process is to construct an argument. This process may include inductive, deductive or abductive reasoning, use of intuitions, making conjectures and testing of them etc. Both cognitive and affective factors influence this process (Furinghetti & Morselli, 2009). It may also include construction of sub-arguments, which are needed in other parts of reasoning. Also, different kind of representations may be used.

Toulmin's (2003) model of argumentation in mind, it can be said that the aim of argumentation is to construct an explanation (*a warrant*) for why the information concerning the initial state (*the data*) necessitates the statement which is argued (*the conclusion*). In some cases, a justification for the authority of the warrant (*backing*) is also needed. The same conclusion can often be argued by using different kind of arguments.

In order to answer to the question why *proofs* are needed in mathematics, the following functions are often presented: *verification, explanation, systematisation, discovery* and *communication* (Hanna, 2000; de Villiers 1999). Argumentation has a broader meaning than the term proof, but it can be thought to have these same functions too. In the following, especially the functions of verification, explanation and systemisation are evaluated with respect different views of mathematics.

DIFFERENT VIEWS ABOUT MATHEMATICS AND ARGUMENTATION

Mathematics can be considered either as *a toolbox*, as *a system* or as *a process* (Törner & Grigutsch, 1994). The toolbox-view means that mathematics is seen as a set of skills, the system-view means that mathematics is a logical and rigorous system and according to the process-view mathematics is a constructive problem solving process (Törner, 1998). Ernest (1989) has presented a corresponding division by defining that *the instrumentalist view* means that mathematics is seen as "an accumulation of facts, rules and skills to be used in the pursuance of some external end" (p. 250), *the Platonist view* means that mathematics is "a static but unified body of certain knowledge" (p. 250), and the *problem solving view*, for one, means that mathematics is "a dynamic, continually expanding field of human creation and invention" (p. 250).

The toolbox/instrumentalist view and argumentation

The view of mathematics has an influence on how teaching and learning of mathematics is seen. On the basis of several studies, Beswick (2005) has connected the instrumentalist view to content-focused teaching in which emphasis is on performances and learning is seen passive reception of knowledge. At least alone, the toolbox/instrumentalist view is neither sufficient nor desirable, because deep understanding, knowledge construction and learners active role have been omitted from it, and mathematics as such is not seen interesting. Therefore, the role of argumentation is only to ensure the correctness of facts and rules, that is, only verification can be considered as a function of argumentation.

The system/Platonist view and argumentation

The Platonist view is often connected to an objectivistic worldview. According to Ernest, the Platonist view involves understanding mathematics as a consistent, connected and objective structure. Mathematical objects are seen to be real and exist independently of human (Brown, 2005). Mathematical statements are considered to be objectively true or false and their truth-value is also seen to be independent from human. In addition, mathematical knowledge is seen to be non-empirical. This kind of objectivistic view of knowledge implies easily that, in the classroom, the teacher is seen as an explainer and the learning is seen as a reception of knowledge. According to Beswick (ibid.), the Platonist view implies content-focused teaching, but, however, the emphasis is on understanding and learning is seen as an active construction of understanding.

On the other hand, the view about mathematics as a system is so salient that it cannot be omitted. The building of an axiomatic system can be seen as an essential goal in mathematics [1]. *Systemisation* means that various known results are ordered into a deductive system, and it has usually been considered as one important function of proof and proving (Hanna, 2000; de Villiers, 1999). According to De Villiers (ibid.), systemisation is useful, because it "helps to identify inconsistencies, circular arguments and hidden or not explicitly stated assumptions" and because it "unifies and simplifies mathematical theories by integrating unrelated statements, theorems and concepts with one another, thus leading to an economical presentation of results" (p. 277). In addition, de Villiers mentions global perspective and easiness in applications as benefits of systematisation.

Mathematics can yet be considered as a consisted and connected structure without any global or objective meaning. It may be seen either as a personal or socially shared construction, which works as a frame of reference in mathematical reasoning. It is not seen as an objective system, but the wideness in which the system is socially shared may vary. This kind of view is well compatible with the problem solving view, too.

If mathematics is considered as an axiomatic system, an important function of argumentation is to connect a statement to the system. Therefore, it is important that the argument is based on the elements of the existing system. By applying Toulmin's model, the concept of *a formal argument* can be defined in the following way:

> An argument is *formal,* if its warrants are based on definitions, axioms and previously proven theorems, i.e. the elements of an axiomatic system.

Usually formal arguments are rigor and detailed, and, thus, they remove all doubts and uncertainty about the truth of a statement. Therefore, in addition to systemisation, verification is their important function too.

The process/problem solving view and argumentation

According to Beswick (ibid.), the problem solving view can be connected to learner-focused teaching, in which learning is seen as autonomous exploration of the learner's own interests. Beswick sees the process/problem solving view to be in accordance with the principles of the constructivist learning views. If mathematics is looked from this point of view, it is important that the learners understand the content conceptually and holistically and that they can connect it to their earlier experiences, either inside or outside the field of mathematics. In addition, invention of new creative ideas is important. According to de Villiers (ibid.), the aim of an explanation is to help an individual to understand the reasons, why a statement is true, in other words, to provide an insight into why the statement follows from the given data. This function of argumentation is crucial, if mathematics is seen from the process/problem solving view.

Weber and Alcock (2004) have presented a categorization for proof production, which contrasts the functions of verification and explanation. According to their original ideas, *a syntactic proof production* refers to reasoning in which inferences are drawn using only symbolic manipulation in a logically permissible way and *a semantic proof production* means that different kind of internally meaningful representations or mental images (Weber and Alcock use the term instantiations) are used to guide the reasoning [2]. Weber and Alcock regard the semantic proof production primarily as a support for the syntactic proof production so that it guides in choosing proper facts and theorems to apply. By the semantic proof production an individual can in a meaningful way make sense of the claim, get suggestions about inferences that could be drawn and become convinced at an intuitive level about the truth of the claim.

A similar contrast between the functions of verification and explanation comes out in Raman's (2002; 2003) categorisation of arguments to private and public ones. According to her *a public argument* has to be sufficiently rigorous for a particular mathematical authority, like a teacher at school, and it has to reveal step-by-step the progress of inference and justifications for each step. Instead, *a private argument* is an argument engendering understanding and having an essential role in facilitating conceptual and holistic understanding of relationships between concepts. According to her, private arguments are often based on empirical or visual data.

Mental imagery is important both in understanding explanations and in their construction. According to Presmeg (2006a), a mental imagery may occur in various modalities, such as sight, hearing, smell, taste or touch, but in mathematical thinking the visual modality is the most prevalent one. Perhaps due to that, the role of visual representations in learning of mathematics and in mathematical reasoning is an issue which has widely aroused interest and vivid discussion among mathematics educators (Presmeg, 2006b).

In order to take the need for explanations as a function of argumentation into account, the following definition for *an informal argument* can be presented:

> An argument is *informal,* if its warrants (cf. Toulmin's model) are based on concrete interpretations of mathematical concepts, which may be based on visual or other illustrative representations.

According to this definition, characteristic to informal arguments is that mathematical concepts are interpreted by using illustrative representations. Perhaps, visual representations are the most important ones in that, but, in addition to them, mathematical concepts can be illustrated, for example, by relating them to some physical context. However, the illustrative effect of a representation may depend on personal experiences, situational factors and the field of mathematics. In the next section, an example concerning the concept of derivative is presented.

FORMAL AND INFORMAL ARGUMENTS

An example: Formal and informal reasoning concerning the concept of derivative

In the following the definitions for formal and informal arguments are illustrated with an example concerning the concept of derivative.

The formal definition of derivative in the case of a real-valued function of a single variable is based on the concepts of function, limit and real numbers. If these concepts have been defined earlier, the above-mentioned definition connects the concept of derivative to the axiomatic system. [3]

By using visual representations, the meaning of the derivative can be described by referring to the steepness of the graph of a function. It can be explained that the sign of the derivative reveals whether the graph is going up or down, and the absolute value describes how steep the uphill or downhill in the graph is. It can also be said that the derivative at a given point is the slope of a tangent line drawn to the graph at this point. If more dynamic visualisation is wanted, the derivative can be illustrated by sliding a pencil along the graph from left to right so that the pencil always lies over the tangent line, and the nib of the pencil points in the direction to which the graph is going at the point in question (Hähkiöniemi, 2006).

An instantaneous rate of change of some quantity can be regarded as a physical interpretation of derivative. For example, instantaneous speed is the derivative of the passed distance, instantaneous acceleration is the derivative of the speed and the electric current is the derivative of the flowing electric charge through a surface.

For example, let us take the theorem stating that the derivative of a constant function is everywhere zero. Starting from the formal definition, this can be shown by a short calculation. Visually, the same result can be reasoned by explaining that because the graph of a constant function is a horizontal straight line, it does not have any uphill or downhill and the tangent drawn to the graph is everywhere a horizontal straight line. Physically, the same thing can be reasoned by explaining that if a quantity is constant, its value does not change, and thus the rate of change is everywhere zero.

Relationship between formal and informal arguments

Above it has been described how formal arguments usually serve the functions of systematisation and verification and informal arguments often serve the function of explanation. However, this categorization is not absolute: A formal argument may be explanative, but this requires that its central ideas above the details are recognised. On the other hand, an informal argument may in some cases be general and rigorous enough so that it is sufficient to verify a statement. Especially, the role of visual arguments has occasionally raised vivid discussion among researchers of mathematics education (Presmeg, 2006b). Several researchers have proposed that visual arguments should be considered as an intended and accepted form of final arguments (Arcavi, 2003; Dreyfus, 1994; Rodd, 2000).

Construction of formal arguments often requires exact and detailed analytic reasoning based on symbolic representations and procedural skills to carry out calculations and other technical procedures. However, informal arguments may reveal holistic features and wider trends, which, yet, may also be very important in the construction process of the argument, by simplifying and concretising the problem situation. Experienced mathematicians are often able to utilize informal elements, like visualization, in an effective way in their reasoning. Stylianou (2002) noticed that mathematicians use visualization in a very systematic way, so that in their reasoning the visual and analytic steps were very closely connected and they interact with each other. Also Raman (2002; 2003) found that mathematicians considered visual and formal arguments closely connected so that the visual arguments in an essential way contributed to inventing ideas in construction of the formal argument. Mathematicians were able to use and construct heuristic/informal and procedural/formal ideas simultaneously so that both ideas clarified each other. Instead, students could not recognise connections between visual and formal arguments. Stylianou's and Silver's (2004) study revealed that mathematicians also saw a wide variety of problems where visualization could be used, whereas students considered visual representations useful mostly in geometrical problems.

The importance of formal and informal arguments is dependent on personal and institutional needs. As well, they may have different roles depending on the field of mathematics. Especially, students may have different tendencies with respect to formal and informal reasoning depending on the field area. Weber and Alcock (ibid.) found that in the case of algebra students' reasoning was too much restricted around the formal definitions of the concepts, but in the case of analysis several studies have reported students' tendencies to use informal approaches without sufficient connections to the formal theory (Juter, 2005; Pinto, 1998; Viholainen, 2006; 2007; 2008; Vinner, 1991).

The use of the division of arguments into formal and informal ones

On the basis of the above-presented definitions, it could be possible to categorize arguments into informal and formal ones. According to these definitions, the decisive difference between formal and informal arguments is in the natures of the warrants. It is not decisive, how much and what kinds of representations have been used in reasoning, when the arguments are constructed. For example, the use of visualization as an aid of thinking in reasoning does not make an argument informal. On the other hand, in construction of an informal argument, the applied visual or physical interpretations may be justified by using formal definitions, but this does not make the argument formal. In this case, the definitions work only as a backing (cf. Toulmin's model) for the used informal interpretations. Therefore, the categorisation of arguments based on these definitions has to be made on the basis of the final forms of the arguments, not on the basis of the reasoning processes. This, actually, distinguishes in an essential way that division both from Weber's and Alcock's division and from Raman's division.

Weber and Alcock see the semantic proof production, as well as Raman sees the private arguments, as important elements of a reasoning process, but not as a final goals. Instead, the definitions of formal and informal arguments make it possible to consider any of them as a final goal of argumentation. No mutual order between the types of arguments with respect to importance etc. follows from the definitions. Therefore, this division provides a framework for design of teaching, especially, for design of argumentation tasks, in which both the need of systematisation and verification based on the view of mathematics as a system and the need of explanation based on the view of mathematics as a problem solving process are equally taken into account. Traditionally, only formal arguments are considered as intended and desirable in mathematics, but if the need of explanation is seriously taken into account in the design of teaching, opportunities to exercise both understanding and producing more explanative arguments should be provided for learners.

CONCLUSION

The presented division of arguments into formal and informal ones can be used as a starting framework in determining what kinds of arguments students should learn to understand and produce. However, this division does not cover all features of mathematical reasoning: It concerns mainly deductive arguments, but it does not cover, for example, inductive and abductive arguments and arguments, whose warrants are based on some authority. These kinds of arguments may have an important role in mathematical reasoning, especially, in the affective level. The broad variety of different forms of reasoning comes out, for example, in Harel's and Sowder's (1998) classification of proof schemes. For example, on the basis of this classification, it could be possible to extend the presented framework.

In addition, it is notable that the nature of mathematics as an axiomatic system comes explicitly out mainly in the tertiary-level. As well, the above-mentioned studies about mathematical reasoning concern mainly the tertiary level. Therefore, more studies about applicability of this division into the lower levels, where the structural nature of mathematics is necessarily not as explicit, would be needed. In addition, the nature and the purpose of informal arguments may differ depending on the field of mathematics. This should be studied more too.

NOTES

1. In 1931, Gödel proved incompleteness theorems, which showed that it is impossible to find a complete and consistent axiomatic system, which includes natural numbers. Due to that, an ambitious attempt to build a complete and consistent axiomatic system including all mathematics proved to be impossible.

2. Alcock and Inglish (2008; 2009) and Weber (2009) have later modified these definitions.

3. In practice, the concept of real numbers is rarely defined properly before presenting the definition for the concept of derivative.

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