

MAKING DISCOVERY FUNCTION OF PROOF VISIBLE FOR UPPER SECONDARY SCHOOL STUDENTS

Kirsti Hemmi

Mälardalen University, Sweden

Clas Löfwall

Stockholm University, Sweden

The paper presents an analysis of a teaching experiment with seven high achieving upper secondary school students in Sweden focusing on the de Villiers' discovery function of proof. The aim of the experiment was to test if it is possible for students to get insights to, and use this function. The data consists of a tape recorded introductory pass, students' group work and the final discussion together with the students. The results show that the students got some insights about the function. However, it was difficult for the students to construct the original proofs in order to use them to discover new results. The paper also shows that the function of discovery needs to be explored and clarified, as there are different interpretations of it in our field.

Key words: mathematical proof, discovery function, transparency, upper secondary school mathematics

INTRODUCTION

Many mathematics educators have explored and discussed the functions of proof in mathematics as science (e.g. conviction, explanation, intellectual challenge) and their relevance for the teaching and learning of proof (e.g. Bell, 1976; de Villiers, 1990; 1999; Hanna 2000; Weber, 2002). Several studies also apply these functions in empirical studies (e.g. Knuth, 2002; Hemmi & Löfwall, 2009). The discovery function was first presented by de Villiers 1990. With this function he refers to discovery/invention of new results by purely deductive manner exploring and analysing a proof.

To the working mathematician proof is therefore not merely a means of a *posteriori* verification, but often also a means of exploration, analysis, discovery and invention. (de Villiers, 1990, p. 21)

New results in his examples often refer to generalisation of the initial statement. The function of discovery as de Villiers defined it has not been so much in the focus of empirical research. Miyazaki (2000) conducted a teaching experiment with tasks specifically designed for students' engagement with activities connected to the discovery function. However, his concept of the function is wider than the one that de Villiers (1990; 1999) presents.

The functions of proof are sometimes interpreted in different ways by researchers. There is also some confusion about the difference between the concept of *function of proof* on the one hand and *approach to proof* on the other hand (i.e. how we work

with proofs and proving and how we present mathematics) (c.f. Hemmi, 2010). For example, Knuth (2002) although referring to de Villiers connects the function of discovery, not to finding new truths by deductions but quite the opposite, inductive ways of finding patterns and making conjectures that may be followed by deductive proofs. Hence, there seems to be a need to scrutinise and clarify also the concept of function in order not to create confusion between different research results and, in the end, their consequences to the teaching practice.

De Villiers (2007) discusses the recent focus on the investigative working manners for example in the mathematics textbooks and points out that they do not promote students' understanding of and skills in using the function of discovery but leads rather to the need for verification, the function that has traditionally been in focus in mathematics education. We do not question that "new" ways to approach proof (investigations-conjectures-proofs) may enhance students' understanding and appreciation of the verification function in another way than just confronting them with complete proofs constructed and verified by others. We also agree with de Villiers (2007) who points out the importance of the balance between experimentation and deductive thought.

The aim of our ongoing study is to develop and test some tasks that could enhance students' understanding of the discovery function as defined by de Villiers (1990). We also take as our starting point that proof sometimes provides valuable insights into why something is true and that looking back and reflecting on it can enable one to generalise or vary the results in different ways (c.f. de Villiers, 2007). In this paper, we present an explorative study that aims to enlighten the function of discovery among some high-achieving upper secondary school students through special tasks designed for this purpose. The earlier studies have mainly focused on geometrical problems. In our study, we explore this function and students' encounter with it within both algebraic and geometrical contexts.

THEORETICAL STANDPOINTS

We look at our experiment from the perspective of social practice theory applied to proof and proving in mathematical practice (Hemmi, 2008) and consider proof as an essential *artefact* in mathematical practice. Artefacts are tools that mediate knowledge between the social and the individual. According to the theory there is a balance between how much to focus on artefacts and how much to work with them without a focus on them, called the *condition of transparency* and a lot of research in our field illuminates this balance in different ways (c.f. Hemmi, 2008). According to the social practice theory, an important part of learning is experiencing meaning in the practice (Wenger, 1998). Understanding the role and functions of proof in mathematical practice could enhance students' experience of meaning in both scrutinising and reflecting on complete proofs and trying to construct their own proofs. Teaching is seen to only offer possibilities for learning and we want to study

what aspects of the object of learning become visible for the students (c.f. Marton & Booth, 1997).

What we mean by the discovery function

The example that de Villiers (1990; 1999) gives in order to enlighten the function of discovery is about a kite where the midpoints of the sides form a rectangle. The perpendicularity of the diagonals is the essential step in the proof and the property of equal adjacent sides is not required. Hence, it was possible to generalise the result to any quadrilateral with perpendicular diagonals. Recently, de Villiers (2007) presents another problem that further enlightens the function of discovery. Miyazaki (2000) extends the meaning of the discovery function to concern not only the generalisations of the original results by using the proof but also finding tacit assumptions, making new mathematical concepts and so on (Miyazaki, 2000, p. 5).

This is a very wide conception about the discovery function and some of the aspects Miyazaki includes in the function (e.g. finding tacit assumptions) have been connected to the function of systematisation for example by de Villiers (1990; 1999).

The aim of our study is to make visible certain aspects of the discovery function for upper secondary students in order to enhance their understanding of them and at the same time enhance their appreciation of proof as a useful tool in mathematical activities. Therefore, we created some special problems the proofs of which could be used for finding new results.

Miyazaki (2000) first let students work with a proving task and analyse the logical structure of their individual proofs carefully. Then, the students received an additional problem where they would study the conditions of the initial problem and find out generalisations. The Swedish students are not as familiar with proving tasks as the Japanese students seem to be according to the results of Miyazaki's study. We also wanted to control the experiment by holding some variables fixed in order to be able to look at more limited aspects of the function than Miyazaki did. Therefore, we focused on the investigations of the initial proofs in order to find out more general statements that the same proof would work for. In Miyazaki's experiment, new proofs were sometimes needed in order to solve the additional problem. In our examples, this is not the case.

The following examples illustrate our view of the discovery function and were used when introducing the students to the topic. We also carefully present the proofs of the examples in order to show the possibilities of using the function in different ways.

Statement 1

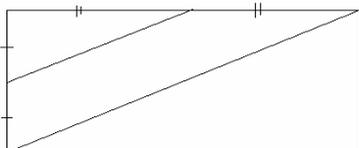
In a rectangle the midpoints of the sides are connected. Then one obtains a parallelogram.

Proof

Draw the diagonal in the rectangle. We will use the following theorem in Euclidean geometry:

Let P be a point on the line AB and Q a point on the line AC. Then PQ is parallel to BC if and only if $AP:PB = AQ:QC$.

It follows that the line connecting the two midpoints on the same side of the diagonal is parallel to the diagonal:



Figur 1

The same is true for the side on the other side of the diagonal. Hence these two lines are parallel. Also, we find that the other two lines are parallel, by considering the second diagonal. *QED*

Analysing this proof, we see that we have not used the fact that the quadrilateral is a rectangle. The proof goes through as it stands for any quadrilateral. Hence, we get a new true statement:

Connecting the four midpoints of the sides of an arbitrary quadrilateral, yields a parallelogram.

In this way we have got a more general result, by realising that the proof for the original statement is valid under weaker assumptions.

It is also possible to obtain new results, by realising that the proof in fact proves more than the original statement. In the next example we will see an application of this. One can combine these two ways to find new truths by both making the assumptions weaker and the conclusion stronger. Formally, this may be illustrated in the following way:

$$A \Rightarrow C \Rightarrow D \Rightarrow B$$

Here $A \Rightarrow B$ is the original statement and $C \Rightarrow D$ is the newly discovered statement.

Another way to create new statements is by “generalisation”. The proof is perhaps a special case of a more general proof. In the above example we see that the proof uses a theorem in the special situation where the proportion is 1:1. The proof works equally well if we instead divide the sides outgoing from two opposite corners in the same proportion. Hence, we get the following generalisation of the original statement:

In a quadrilateral the two sides outgoing from a corner are divided in the same proportion. The same proportion is also used to divide the two sides outgoing from the opposite corner. Connecting these four points gives a parallelogram.

Next, we illustrate what we mean by the discovery function with an algebraic example.

Statement 2

If two prime numbers greater than 2 are added, then the result is not a prime number.

Proof

A prime number greater than 2 cannot be divisible by 2. Hence it is 1 greater than an even positive number. If two such numbers are added, the result is a number which is 2 more than an even positive number. This number is greater than 2 and divisible by 2 and hence it is not a prime number. *QED*

Analysing the proof, we see the following structure:

x, y prime numbers $> 2 \Rightarrow x, y$ odd $> 1 \Rightarrow x+y$ even $> 2 \Rightarrow x+y$ not prime

We may hence discover a new truth by both weaken the assumptions and draw a stronger conclusion:

$$x, y \text{ odd } > 1 \Rightarrow x+y \text{ even } > 2$$

Compare with the general picture above, $A \Rightarrow B$ is replaced by $C \Rightarrow D$. Finally, we remove the assumption, $x, y > 1$, and weaken the conclusion to, $x+y$ even, to obtain a more aesthetic statement. Hence, we have discovered the following truth by examining the proof.

The sum of two odd numbers is even.

Is it possible to find generalisations?

Here is one possible generalisation (with the same proof):

If the integers a and b , when dividing them with the integer k , have the rests r and s and $r+s$ is divisible by k then also $a+b$ is divisible by k .

The following two problems were left to the students to work with, in two groups during about one and a half hour.

Problem 1

Let n be an integer which is not divisible by 3. Prove that $n^3 - n$ is divisible by 3.

Problem 2

Two circles intersect at the origin of an orthogonal coordinate system. The centre of one of the circles is on the x -axis, while the centre of the other circle is on the y -axis.

The circles intersect at one more point. Prove that they intersect there under right angle.

METHODOLOGY

We introduced the function of discovery to a group of high-achieving upper secondary school students, two girls and five boys. The students were to finish their secondary level studies during the time of the experiment. We chose these students because they had taken some special courses in mathematics, for example in geometry, so we could be sure that they were familiar also with geometrical proofs. In Sweden, not much time is usually spent on geometry in ordinary upper secondary school classes.

During the introduction, we presented the examples above. Clas led the session and engaged also the students in the presentation with appropriate questions. The students did not need to take notes because we handed out the written presentation to them after the introduction. Also their mathematics teacher took part of this session together with the students. Kirsti observed the session and she also videotaped the presentation. The presentation took about 30 minutes.

After the presentation we divided the students into two small groups according to the recommendations of their teacher and they obtained the written introduction with the two tasks that they would work with. We asked the students to read the tasks individually and then together discuss and try to solve them. They could come and ask for help if needed and Clas visited the groups twice during the session in order to offer his help. The group work was tape-recorded.

After the groups had struggled with both problems about one and a half hour we gathered the students together again and asked them to tell us how they had solved the problems. This discussion was also tape-recorded. We kept also the notes that the individual students had made during the group work session in order to use them as a complementary data. We decided to meet after one week and the students could read the material we handed out to them and think about two additional tasks.

During the last meeting we asked the students to tell us if they had obtained some new insights concerning the use of proof and proving. Finally, Clas showed the solution of one of the tasks they had been thinking about at home. This session was tape-recorded as well.

Data analysis

We watched and listened the videotaped session to find out students insights during the presentation. Then, we listened the tape-recorded group work sessions several times and identified the parts that in various ways enlightened the students' insights concerning the discovery function. We transcribed the relevant parts of the discussions. In parallel, we also studied the notes that the individual students made during the group session.

Ethical aspects

We carefully informed the students and the teacher about the background and the aim of our study and that it was optional to take part of it. We informed also what we would do with the data and that the data would be handled in a way that would protect their anonymity.

RESULTS OF THE TEACHING EXPERIMENT

In the presentation of the Statement 1 above, some students nicely pointed out that one even obtains a rhombus and proved that the sides are equal using the Pythagorean Theorem. However, to be able to use the discovery function, Clas showed them the proof given above. Concerning the algebraic example, one of the students seemed to catch the idea of discovery already and suggested a generalisation of the statement.

The analysis of the group sessions shows that students in both groups seemed to realize that it was a question about finding new results. Group 1 started their work enthusiastically encouraging each other:

S1: Now, let's find new truths!

They also show that they have understood that it is important to first find a solution to the original problem first.

S2: First, we have to show that we have a solution.

However, they did not manage to find a proof for the initial statement which was a prerequisite for exploring and deriving deductively new results. Instead, they started to discuss the possibilities of proving something more general.

S1: Is it possible to take away some demands in some way?

S3: I wonder if it works for all odd integers.

The second group managed to prove the statement by expressing the number n with $t+1$ and $t+2$ where t was divisible with three. During the proving process they also noticed that the expression is always even. However, they did not succeed to put their results together and extend the divisibility result from 3 to 6. They did not notice either that they could omit the prerequisite that n is not divisible by three. In the similar manner as the first group they started to make and test conjectures about possible generalizations without reflecting on their proof.

S4: But the point is that this is a correct proof and we have to go on and what was it we would do we would generalise or specify...why not just put $n^x - n$ divided by x ?

Then they tested the generality of the statement and arrived at a conjecture that $n^x - n$ is divisible by x if x is a prime. Hence, they did not reflect on the proof they had

constructed but went on inductively testing new conjectures inspired by the initial statement.

Some students suggested a proof by induction for the first problem and also worked a bit with it. However, they did not seem to understand the idea of mathematical induction properly.

S5: The proof by induction, would it work here?

S4: We can always do it.

S6: But I don't remember proof by induction.

S4: But exactly like we have started now (testing with the conjecture) we start by the basic and then you know one first proves that it holds for n and then that it works for $n + 1$.

S7: Exactly.

One student also suggested a proof by contradiction and here also we noticed that the student did not really grasp the idea from the logical point of view.

S4: Shall we do this classical that we assume the opposite that n is divisible by three and then look how it works?

Regarding the geometry task, both groups managed to prove the initial statement and also to generalize the statement (the angles are equal in both intersection points).

When we met the students after one week and posed a question about what they had learned the students answered:

S5: Mm yes the key word is I think generalise in these proofs and then it is we have learned or any way I have learned I think in another way and it is is that if one can extend the proof and use it and if it works with other assumptions.

S4: The way in which I understood this was that one either generalise or specify that one as you (Clas) expressed it strengthens the prerequisites of this proof or that one opens it to see what more cases hold.

Hence, these students state they had got insights in the aspects of discovery function that we aimed with our experiment. Yet, the second extract shows that the student has difficulties to distinguish between assumptions and conclusions. We also noticed that the two problems they would think about at home were too difficult in the light of the analysis of the group work.

CONCLUSIONS AND DISCUSSION

The students in our study showed in several ways that they caught the idea of the discovery function. However, they had great difficulties to construct and analyse their own proofs in a way the students in Miyazaki's (2000) study did. Miyazaki stresses that it is important for the teacher to let students first themselves prove the

initial statement, in order to successfully apply the function in the teaching of lower secondary school mathematics. Concerning the Swedish students we found out that it could be better to also work with complete proofs in order to engage the students in exploring the very deductions in searching for further discoveries. One of the hindrances for the groups concerning the first problem was the construction of the initial proof.

Our focus was not on the very construction of the proof of the initial statement but on the use of it in order to enlighten the discovery function. If the teacher has designed a situation that aims to enhance students' understanding of certain aspects of proof, it can be better to hold some other aspects constant and vary the ones that are the object of learning (c.f. Marton & Booth, 1997; Hemmi, 2008). Recent international comparisons and national evaluations show that Swedish students are not very strong in algebra (e.g. The Swedish National Agency of Education, 2009; Brandell et al, 2008). This is confirmed by our study. Although the students were considered as high-achieving they had difficulties with some elementary algebra, e.g., finding the factorisation $n^3 - n = n(n+1)(n-1)$. The students in our study had also difficulties to cope with some logical aspects involved in proving. Hemmi's (2008) study shows that many university students still struggle with them.

Our aim is to go on conducting a new teaching experiment where we will modify the design of it according to the results from this study. The present study shows that there are a lot of interesting aspects to explore concerning the function of discovery, both theoretically and in applying it in the teaching of mathematics at different levels.

REFERENCES

- Bell, A. W. (1976). A study of pupils' proof-explanations in mathematical situations. *Educational Studies in Mathematics*, 7, 23-40.
- Brandell, G., Hemmi, K., & Thunberg, H. (2008). The widening gap – A Swedish perspective. *Mathematics Education Research Journal*. The Special Issue on the Transition from School to University Mathematics.
- De Villiers, M. (1990). The role and function of proof in mathematics. *Pythagoras*, South Africa, 23, 17-24.
- De Villiers, M. (1999). The role and function of proof with Sketchpad. In M. de Villiers (Ed.), *Rethinking Proof with Sketchpad* (pp. 3-10). Emeryville, CA, USA: Key Curriculum Press.
- De Villiers, M. (2007). An example of the discovery function of proof. *Mathematics in School* (36) 4, pp. 9-11.

- Hanna, G. (2000). Proof, explanation and exploration: an overview. *Educational Studies in Mathematics*, 44, 5–23.
- Hemmi, K. (2008). Students' encounter with proof – the condition of transparency. *ZDM – The International Journal on Mathematics Education. The Special Issue on Proof*, 40, 413-426.
- Hemmi, K. (in press). Three styles characterising mathematicians' pedagogical perspectives on proof. *Educational studies in mathematics*.
- Hemmi, K., & Löfwall, C. (2009). *Why do we need proof*. A paper presented at CERME 6 - Sixth Conference of European Research in Mathematics Education. Lyon, France. Jan 28th - Feb 1st, 2009.
- Knuth, E. J. (2002). Teachers' conceptions about proof in the context of secondary school mathematics. *Journal of Mathematics Teacher Education*, 5(1), 61-88.
- Marton, F., & Booth, S. (1997). *Learning and awareness*. New Jersey: Lawrence Erlbaum Associates.
- Miyazaki, M. (2000). What are essential to apply the “discovery” function of proof in lower secondary mathematics? In T. Nakahara & K. Mastaka (Ed.). *Proceedings of the 24th Conference of the International Group for the Psychology of Mathematics education*. Vol. 4, 1-8. Hiroshima University. Japan.
- The Swedish National Agency for Education. (2008). Svenska elevers kunskaper i TIMSS Advanced 2008 och 1995. En djupanalys av hur eleverna i gymnasieskolan förstår centrala begrepp inom matematiken. (Swedish students' knowledge and skills in TIMSS Advanced 2008 and 1995. A deep analysis about how upper secondary school students understand central notions in mathematics). Available [2010-08-31] at <http://www.skolverket.se/sb/d/3535/a/7757>.
- Weber, K. (2002). Beyond proving and explaining: Proofs that justify the use of definitions and axiomatic structures and proofs that illustrate technique. *For the Learning of Mathematics*, 22(3), 14-17.
- Wenger, E. (1998). *Communities of Practice*. Cambridge: Cambridge University Press.