MATHEMATICAL PROVING ON SECONDARY SCHOOL LEVEL I: SUPPORTING STUDENT UNDERSTANDING THROUGH DIFFERENT TYPES OF PROOF. A VIDEO ANALYSIS

Esther Brunner, Kurt Reusser, Christine Pauli

University of Teacher Education Thurgau, Kreuzlingen CH, University of Zurich, CH

SUMMARY

Within the framework of the Swiss-German study "Unterrichtsqualität, Lernverhalten und mathematisches Verständnis" ["Instructional Quality, Learning Behaviour and Mathematical Understanding] (Klieme, Pauli & Reusser, 2006, 2009) and by using the example of a purely mathematical problem, it was examined in 32 classes how teachers support the process of proving in classroom instruction from a subject-based and a communicative point of view. For this purpose, an analysis instrument was developed which describes content-related aspects of the problem-solving process as well as the students' participation. The results clearly indicate that the individual teachers differ in terms of their choice and application of specific types of proof. A special group, however, is constituted by those teachers who prove in multiple ways.*

Keywords: Mathematical proving, Mathematics instruction, Secondary school level I, Support of the students, Video analysis

INTRODUCTION

In the context of educational standards (cf. NCTM, 2000; Blum et al., 2006; EDK, 2010), learning to prove and argue has gained new significance and experiences a real renaissance. This change was particularly furthered after the criticism of formal proof and its strictness could be neutralized with respect to public school instruction and was complemented by other concepts like, for example, pre-formal (or operative) proving (e.g. Krauthausen, 2001).

Since mathematical proving is a demanding activity, it requires teachers to support their students in a way which is close to contents and understanding-oriented. And since a mathematical proof is accepted or rejected by the community, argumentation takes place within a discourse. For these reasons, two aspects are crucial to the support of proving: content-related support as well as participation in technical discourse. This paper is focused on content-related, technical support.

Various empirical studies have shown that rather few students are able to give mathematical reasons for or to prove a given fact (cf. Healy & Hoyles, 1998; Reiss, Klieme & Heinze, 2001). As regards geometrical proofs, Reiss and collaborators (Reiss, Hellmich & Thomas, 2002) found that the varying capabilities of students in

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terms of proving can be substantially explained by class membership. Against this background it is surprising that only relatively few studies focus on the part of the teacher and his or her support behaviour in proving.

THEORETICAL FOUNDATIONS

Proving as an Important and Demanding Mathematical Core Activity

The debate around proving in mathematics instruction is multi-faceted and extensive. Besides questions about the strictness of a proof (cf. Hanna, 1997; Jahnke, 1978) or its significance (cf. Heintz, 2000), there is an interest in the different functions of a proof (cf. Hanna & Jahnke, 1996; Hersh, 1993; de Villiers, 1990), or in the role of communication and the community in proving. The distinctions between process and product, or proof and proving as an activity are likewise objects of the debate. In a genetic conception of learning, the process of proving is important especially to mathematics instruction. It is for this reason that Jahnke (1978), by reverting to Freudenthal’s proposition of “local ordering” (1977), recommends that students should also be offered an understanding of sufficient reasons while problem-solving. It is not the completed proof alone or its re-enactment that should dominate mathematics instruction. Various didactic models of teaching proving processes are based on this assumption (cf. Boero, 1999; Reiss, 2002). By starting out by a (constructed) need for proving (cf. Hefendehl-Hebeker & Hussmann, 2003; Reusser, 1984), these models suggest different steps of how to get from a “why”-question to argumentatively supported reasoning. This reasoning, however, must turn out to be conclusive enough also for other persons, and it must be possible to validate it. Owing to this validity claim and external validation, a proof obtains a communicative function as well. Correspondingly, didactic communication and together with it orality are of high importance in classroom instruction.

Types of Proof

As various authors highlight the importance of sufficient reasons in connection with the strictness of a proof, it becomes clear that in mathematics instruction not only proof in the mathematically strict sense but also its action- and thought-psychological precursors in the sense of more or less “strong” reasoning (“pre-formal proof”) should be permitted.

There are numerous different classifications of proofs (cf. Balancheff, 1988; Blum & Kirsch, 1989; Leiss & Blum, 2006). The classification of Wittmann and Müller (1988) constitutes the basis for this study. It differentiates between three fundamentally different types of proof: 1) formal-deductive proof, 2) experimental proof, and 3) content-related illustrative proof.

Formal-deductive proofs are based on the logical deduction of a statement which follows from another statement step by step. They are related to conciseness of formulation which is manifested in as brief a form as possible, i.e. the mathematical formula or the formal language. Because of this, formal-deductive proofs require not only a particular way of thinking but also a particular procedure backed by a specific technical terminology.

In contrast to formal-deductive proofs, experimental proofs do not yield conclusive certainty and are hence not considered as proofs in the strict sense, but remain tied to the respective examples. Yet working directly with examples is fruitful especially for younger and less competent students, and it facilitates argumentations which are bound to examples. Moreover, experimental approaches are suitable for generating a subjective need for proving.

As far as content-related illustrative proofs are concerned, formalism is only of limited significance as well, since reasons should not solely be derived by showing plausible examples as this is the case with experimental proofs. Content-related illustrative proofs are rather based on constructions and operations which render it intuitively discernible that they are applicable to a whole class of examples, and that certain conclusions can be drawn (cf. Wittman & Müller, 1988, p. 249). It must be possible to generalise content-related illustrative proofs directly from the given example. This process of generalisation should be as intuitively discernible as possible which is facilitated by making the mathematical structure transparent, mostly on the enactive or iconic level.

These three types of proof each make different demands on the students' competences, and at the same time they offer different approaches – and as a consequence specific, content-related support – depending on the current level of knowledge of the class. A genetic procedure might start out by an experimental or content-related illustrative proof and then advance to a formal-deductive proof, thus inducing an increasing degree of abstraction and symbolization.

Research Questions and Aims of the Study

The aim of the current analysis is to describe teachers' support in phases of mathematical proving in secondary school level I instruction as extensively as possible and to analyse it comparatively. The following two research questions are guiding the study presented below:

- Which types of proof can be observed while the proving and reasoning problem is being worked on?
- Is it possible to distinguish special groups of teachers whose support in proving phases differs clearly from the support of the majority?

METHOD

In order to compare instructional processes on the micro-level, it is necessary to fall back on standardised contents. This is realised by the data set of the video-based study of mathematics instruction in secondary school level I classes named "Unterrichtsqualität, Lernverhalten und mathematisches Verständnis" ["Instructional Quality, Learning Behaviour and Mathematical Understanding] (Klieme, Pauli &

Reusser, 2006, 2009), also known as the “Pythagoras Project”, which ran from 2000 to 2006.

In total, 5 lessons per teacher on two different thematic units were recorded: one unit on the introduction to the Pythagorean Theorem, and one unit on solving mathematical word problems picked from a given set. One of these problems was a purely mathematical proving and reasoning task. This paper refers to the handling of this task, with the content-related work in class while reasoning and proving being at the centre of the analyses.

Data Set

The sample consisted of 32 classes of the 8^{th} or 9^{th} year from the highest track (“Gymnasium”) and from the middle track (“Realschule”/”Sekundarschule”) of the three school types in Germany and Switzerland.

In order to enable as representative an insight as possible into the everyday handling of a reasoning problem, the teachers were not specifically prepared. Several days before the videotape session, they were sent a set of word problems which had to occur in the instructional unit. Apart from that, the teachers could freely decide on the methodological-didactic arrangement.

For the analysis presented in the following, the provided proving and reasoning task was selected. The object of the analysis is the complete tackling of the problem. This was realised in classroom discourse but also in the form of learning support in phases of group work and silent work.

Mathematical Problem

The task in question is a purely mathematical proving and reasoning problem which opens up different procedural options.

The sum 13 + 15 + 17 + 19 is divisible by 8. Does this hold for any sum of four ensuing uneven numbers?

The given fact can be both elucidated by means of a numerical example (experimental and/or content-related illustrative proof) and worked out through a formal-deductive procedure. As the formal-deductive procedure is not too demanding for the age group in focus, the students can actively participate in the line of argument when they are appropriately supported.

Instrument

To be able to describe proving phases in a differentiated manner, in a first step a subject-didactic analysis instrument was developed by means of which the 32 cases were coded. In total, 117 different features of the way of working on the problem were captured per case. These features constituted the starting point for further analyses.

The instrument, specially developed for this analysis, captures features both on the surface structure and on the deep structure of instruction (cf. Reusser, 2005). The three big domains of the instrument refer to 1) the context of working on the problem, 2) the communicative dimension and 3) the content-related dimension. The context of working on the problem captures, among others, the duration and the arrangement pattern of working on the problem. The communicative dimension captures features like the conceptual level, communication of meta-rules and meta-commentaries, and others, as well as the type of didactic communication and the students' participation for every content-related feature. The content-related dimension captures the type of proof chosen by the teacher, the heuristics made use of, the elements of comprehension as actual content-related kernel of the problem, as well as rather peripheral features of working on the problem.

The current paper is restricted to the presentation of results referring to the type of proof made use of.

Video Analyses

The 32 instructional units were coded with the help of the analysis instrument (inter-rater reliability: 88–100%). The thus yielded data could then be further processed by means of statistical analyses. The first results, presented below, are based on descriptive statistics and on group comparisons of individual procedural steps.

RESULTS

Type of Proof

All of the three different categories of proof occurred, though in a quite varying distribution. Formal-deductive proofs could be found in 65.5% of all cases (21 classes), experimental proofs in 12.5% of the cases (4 classes), and content-related illustrative proofs in 37.5% of the cases (12 classes). Consequently, there are classes in which several categories of proof could be observed. This was true for 9 classes. In 4 classes, however, no proof at all was (conclusively) completed. The choice of the type of proof does not bear any connection with the type of school.

Multiple Proving – A Special Group

In total, 9 classes could be determined in which two of the three types of proof were implemented. These classes can be described as a special group because the use of different types of proof can be regarded as an elaborated line of argument.

That these classes constitute a special group can be inferred from statistically significant differences. These teachers clearly made more use of heuristic aids in their classes ($M = 2.33$; $SD = .86$; $N = 9$) than those teachers who only realised one type of proof ($M = 1.53$; $SD = .96$; $N = 19$). This difference in the mean is statistically significant ($t = 2.13$; $df = 26$; $p < .05$), and it displays a very strong effect ($ES_d = .86$).

Statistically significant differences between classes that carry through several types of proof ($n_1 = 9$), those that make use of one type of proof ($n_2 = 19$), and those that do not implement any proof ($n_3 = 4$) can also be detected with respect to other features in the reasoning process, for example concerning the formal-deductive procedure ($\chi^2 =$

9.55; df = 2; $p < .01$), the content-related illustrative procedure ($\chi^2 = 9.34$; df = 2; $p < .01$), the criterial assessment of a proof ($\chi^2 = 10.38$; df = 2; $p < .01$), and others more. All differences appear in favour of the group which implements several proofs – i.e. in this group, the supportive aids mentioned above (heuristics) are made use of more often than in the other two groups. These classes can therefore be characterised as a special group also on the level of individual features of the solving process.

Since the use of the two different types of proof differs quite obviously within this group, further sub-groups can be differentiated.

Group A: Experimental and formal-deductive: Two classes of the middle track ("Sekundarschule"/"Realschule") belong to this group. Both classes first carry through an experimental proof and discuss its limits. Afterwards, a formal-deductive proof is worked out in a questioning-developing classroom dialogue. Both cases can be regarded as genetic procedures, because a formal-deductive approach is developed out of an experimental approach and the determination of its weaknesses.

Group B: Content-related illustrative and formal-deductive: In six classes there is first a content-related illustrative proof and then a formal-deductive proof. Five of these classes are of the highest track ("Gymnasium"), one is of the middle track ("Sekundarschule"). In one class, the teacher initially asks his students to work out a proving procedure themselves in groups during quite a long period of time. Thereafter, students from two different groups present two different types of proof – first a content-related illustrative proof and then a formal-deductive one – as results of their independent work. Subsequently, these results are discussed and deepened in a classroom dialogue. The other five teachers make use of short phases of independent student work, but they work out the line of argument together with their students by way of a classroom dialogue. The content-related illustrative proof occurs in all classes before the formal-deductive one. In the five classes without presentation of the students' own solutions, the content-related illustrative proof is raised on the next level of generalisation in a further step and the formal-deductive proof is developed. Also this case can be considered as a genetic procedure. It is, however, more structured and more systematic than in group A. In group B, it is not only the weakness of the procedure which is demonstrated by means of the content-related illustrative proof. Moreover, a way of preparatory thinking and a didactic guidance to formal-deductive proving on the basis of the now transparent and understood inherent structure is presented.

Group C: Experimental and content-related illustrative: In one class of the highest track ("Gymnasium") an experimental and a content-related illustrative proof occur. This class works out the two types of proof in a questioning-developing classroom dialogue with the teacher collecting the students' suggestions regarding the procedure. In doing so, he first carries through an experimental proof together with his class and then demonstrates that this proof is limited in its validity claim. Afterwards, he applies a content-related illustrative proof to a concrete example. Also in this case we are dealing with a genetic procedure, though in a different way. The experimental approach is discussed with regard to its weaknesses, and thereafter it is complemented by a content-related illustrative proof.

DISCUSSION

It is not very surprising that in the vast majority of classes a formal-deductive proof is implemented. This can be put down to the fact that strictness of a proof is characteristic of mathematical proving. What is more astonishing is the fact that the use of the type of proof does not depend on the type of school. Formal-deductive proof does not occur more often in classes of the highest track ("Gymnasium") than in classes of the middle track ("Sekundarschule"/"Realschule"), although it is more demanding than the other two types. It was expected that the most ambitious school type would make use of a more demanding procedure than the school type belonging to the middle track. This dominance of formal-deductive proof in both types of school indicates that teachers think that proving is necessarily tied up with a strictly formal-deductive procedure. For this reason, the potential of the different types of proof with regard to specific, content-related support is still underused to a great extent.

It is interesting that a special group can be characterised that differs from the other classes in their use of multiple proving and thus in various supportive features. As far as this group is concerned, it can be assumed that the instruction with respect to proving and reasoning displays a clearly higher quality than in the other classes. The reason for this assumption is that different approaches to a given fact are opened up, and that the focus is not one-sidedly on formal-deductive proving, thus preventing a hasty reference to the strictness of the proof. Rather, we are dealing with a genetic procedure which implies a real guidance to formal-deductive proving by means of a supportive arrangement of the argumentation. This is why in the training of mathematics teachers it should be carefully made sure that they do not only know about the strictness of a proof, but also, at the same time, that they are able to regard this formal strictness as an aim of instruction and not as a prerequisite for the implementation of a proof.

However, the chosen type of proof alone does not state anything about the quality of the understanding-oriented support. Rather, it solely describes a predominant practice. In order to investigate the quality of the understanding-oriented support and of the fostering of argumentation there is need for further in-depth analyses which are currently undertaken. It is examined, among other things, whether and under which circumstances the choice of a certain type of proof is functional, to what extent this choice has effects on the students, and which impact the teachers' beliefs have in this respect. Moreover, further content-connected aids by means of which teachers support their students in understanding and arguing are of interest. Finally there is an interest in the way of student participation while arguing and proving. These analyses are under way as well.

REFERENCES

Balacheff, N. (1991). Benefits and limits of social interaction: The case of teaching mathematical proof. In A. Bishop, S. Mellin-Olsen & J. Van Dormolen (Eds.), *Mathematical knowledge: Its growth through teaching* (pp. 175-192). Dodrecht: Kluwer Academic Publisher.

Blum, W., Drüke-Noe, C., Hartung, R. & Köller, O. (Eds.). (2006). *Bildungsstandards Mathematik: konkret. Sekundarstufe I: Aufgabenbeispiele, Unterrichtsanregungen, Fortbildungsideen*. Berlin: Cornelsen Scriptor.

Blum, W. & Kirsch, A. (1989). Warum haben nicht-triviale Lösungen von f' = f keine Nullstellen? Beobachtungen und Bemerkungen zum inhaltlich-anschaulichen Beweisen. In H. Kautschitsch & W. Metzler (Eds.), *Anschauliches Beweisen* (pp. 199-209). Vienna: Hölder-Pichler-Tempsky.

Boero, P. (1999). Argumentation and mathematical proof: a complex, productive, unavoidable relationship in mathematics and mathematics education. *International Newsletter on the Teaching and Learning of Mathematical Proof*, 7/8.

EDK Generalsekretariat. (2010). *Basisstandards für die Mathematik.* Unterlagen für den Anhörungsprozess. 25. Januar 2010. Bern: EDK.

Freudenthal, H. (1977). *Mathematik als pädagogische Aufgabe.* Volumes 1 & 2. Stuttgart: Klett.

Hanna, G. (1997). The ongoing value of proof. *Journal für Mathematikdidaktik, 18* (2/3), 171-185.

Healy, L. & Hoyles, C. (1998). *Justifying and Proving in school mathematics. Technical Report on the Nationwide Survey.* London: University of London, Institute of Education.

Hefendahl-Hebeker, L. & Hussmann, S. (2003). Beweisen – Argumentieren. In T. Leuders (Ed.), *Mathematikdidaktik. Praxishandbuch für die Sekundarstufe I und II* (pp. 93-106). Berlin: Scriptor.

Heintz, B. (2000). *Die Innenwelt der Mathematik. Zur Kultur und Praxis einer beweisenden Disziplin.* Vienna: Springer.

Hersh, R. (1993). Proving is convincing and explaining. *Educational Studies in Mathematics, 24* (2), 389-399.

Jahnke, H. N. (1978). *Zum Verhältnis von Wissensentwicklung und Begründung in der Mathematik. Beweisen als didaktisches Problem.* Bielefeld: Institut für Didaktik der Mathematik.

Klieme, E., Pauli, C. & Reusser, K. (2006). *Dokumentation der Erhebungs- und Auswertungsinstrumente zur schweizerisch-deutschen Videostudie "Unterrichtsqualität, Lernverhalten und mathematisches Verständnis"* (3 Volumes). Materialien zur Bildungsforschung, Band 13-15. Frankfurt a.M.: dipf.

Klieme, E., Pauli, C. & Reusser, K. (2009). The Pythagoras Study. In T. Janik & T. Seidel (Eds.), *The power of video studies in investigating teaching and learning in the classroom* (pp. 137-160). Münster: Waxmann.

Krauthausen, G. (2001). “Wann fängt das Beweisen an? Jedenfalls, ehe es einen Namen hat.” Zum Image einer fundamentalen Tätigkeit. In W. Weiser & B. Wollring (Eds.), *Beiträge zur Didaktik der Mathematik für die Primarstufe. Festschrift für Siegbert Schmidt* (pp. 99-113). Hamburg: Dr. Kovac.

Leiss, D. & Blum, W. (2006). Beschreibung zentraler mathematischer Kompetenzen. In W. Blum et al., *Bildungsstandards Mathematik: konkret. Sekundarstufe I: Aufgabenbeispiele, Unterrichtsanregungen, Fortbildungsideen* (pp. 33-50). Berlin: Cornelsen.

NCTM. (Ed.). (2000). *Principles and standards for school mathematics.* Reston: NCTM (National Council of Teachers in Mathematics).

Reiss, K. (2002). *Argumentieren, Begründen, Beweisen im Mathematikunterricht.* Projektserver SINUS. Bayreuth: Universität.

Reiss, K., Hellmich, F. & Thomas, J. (2002). Individuelle und schulische Bedingungsfaktoren für Argumentationen und Beweise im Mathematikunterricht. In M. Prenzel & J. Doll (Eds.), *Bildungsqualität von Schule: Schulische und ausserschulische Bedingungen mathematischer, naturwissenschaftlicher und überfachlicher Kompetenz. 45. Beiheft der Zeitschrift für Pädagogik* (pp. 51-64). Weinheim: Beltz.

Reiss, K., Klieme, E. & Heinze, A. (2001). Prerequisites for the understanding of proofs in the geometry classroom. In M. van den Heuvel-Panhuizen (Ed.), *Proceedings of the 25th Conference of the International Group for the Psychology of Mathematics Education Vol. 4* (pp. 97-104). Utrecht: Utrecht University.

Reusser, K. (1984). *Problemlösen in wissenstheoretischer Sicht. Problematisches Wissen, Problemformulierung und Problemverständnis*. Dissertation (reprinted 1994). Bern: University.

Reusser, K. (2005). Problemorientiertes Lernen – Tiefenstruktur, Gestaltungsformen, Wirkung. *Beiträge zur Lehrerbildung, 23* (2), 159-182.

Villiers, M. de (1990). The Role and the Function of Proof in Mathematics. *Pythagoras, 24*, 17-24.

Wittmann, E. C. & Müller, N. (1988). Wann ist ein Beweis ein Beweis? In P. Bender (Ed.), *Mathematikdidaktik – Theorie und Praxis. Festschrift für Heinrich Winter* (pp. 237-258). Berlin: Cornelsen.