ARGUMENTATION AND PROOF:
DISCUSSING A "SUCCESSFUL" CLASSROOM DISCUSSION

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This paper concerns a discussion in a university course of primary school teacher education. The discussion was aimed at developing awareness about the condition of "equally likely possible cases" in the classical definition of probability. During the discussion students engaged several times in reflections about validity of statements, validity of inferences, etc. The aim of this paper is to present some tools to analyze the discussion and hypothesize possible follow-ups, related to the performed analysis and the aim of developing awareness about the "rules" of argumentation and proof.

Keywords: argumentation; mathematical proof; rationality; meta-mathematics

INTRODUCTION

The development of students' awareness about the "rules" of argumentation and proof in mathematics is one of the main challenges for mathematics education. This statement expresses a conviction largely shared among mathematics educators in the last three decades, in spite of different positions concerning *when* to develop such awareness, *which elements* should awareness consist of, and *how* to deal with them in the classroom (cf Balacheff, 1987; Duval, 1991; Harel, 2008).

Concerning the *how*, we think that the rules of argumentation and proof cannot be taught as a separate subject in the phase of approach to them (obviously it can become a subject for specialized courses in more advanced education). For us the best didactical choice is to exploit *suitable mathematical activities* of argumentation and proof, and develop awareness of the rules according to the *occasions* offered by those activities. But what mathematical activities are *suitable* to offer the expected *occasions*? And *how* to exploit those occasions?

In this paper we will deal only with the *occasions* and the *how* question. After a description of the educational context in which the discussion reported in this paper took place, we will try to frame the analysis of the discussion in order to identify the *occasions* offered in it. We will also discuss *how* to exploit those *occasions*.

THE DOCUMENT IN ITS EDUCATIONAL CONTEXT

In Italy, since 1999 prospective primary school teachers must follow a four-year university preparation, including courses, laboratories and teaching stages (in the future a fifth year will be added). At present, students must follow at least four courses of 30 hours each in the mathematical area, which should integrate the revision of basic mathematical subjects together with didactic considerations, with an eye to national indications for primary school curricula. At the Genova University our courses are strictly co-ordinated. Teaching is organized according to a cyclic structure. In most cases, the starting point of each cycle is an individual problem solving activity; it exploits knowledge supposed as shared by students (sometimes recapitulated by the teacher), and/or new knowledge introduced by the teacher, and/or a document coming from primary school classes. A collective discussion leaded by the teacher follows; it concerns students' individual productions, selected by the teacher. Discussion results in a synthesis, which can provide elements for a further cycle - or constitute an end point for the subject at stake. Some contributions of the teacher during and after the discussions concern the historical origin of the mathematical content dealt with in that moment, and reflections on the relationships mathematics - "reality" and on the ways of reasoning in mathematics. Students' interventions and questions on the above issues are encouraged. Traces of the effects of these choices are in the document.

Students attending our courses come from different kinds of high school; the majority comes from a socio-pedagogical high school, with few hours of mathematics (3 or 2 hours each week, according to the grades); some students come from scientific high school (3-5 weekly hours of mathematics) or technical high school (3-4 hours of mathematics, but no mathematics in the last year). In the year 2004/05 49 students attended the fourth course of 30 hours (I was the teacher); 12 hours were dedicated to a technical and didactical introduction to elementary probability theory. 45 students took part in the activity described below. 15 of them (P-students) had already met elementary probability theory in the high school. Like every year, the starting point of the activity on probability was the following individual problem:

> "If we cast two dice and sum the digits on their superior faces, is it more convenient to bet even, odd, or is it indifferent?"

Four solutions were considered for further discussion:

> A) - it is more convenient to bet even, because the possible results are eleven (2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12) and 6 results out of 11 are even (14 answers similar to this, 6 of them are by students who studied elements of probability in the high school: P-students);
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> B) - it is more convenient to bet even, because even+even=even, odd+odd=even, even+odd=odd, thus in two cases out of three the result is even (7 answers, 1 from a P-student)
>
> C) - it is indifferent because even+even=even, odd+odd=even, even+odd=odd, odd+even=odd, therefore two results out of four are odd and two results are even (11 answers, 3 from P- students);
>
> D) - it is indifferent because if I take an even digit of the first dice, like 2, the sums with the digits of the other dice (2+1, 2+2, 2+3, 2+4, 2+5, 2+6) are even in three cases and odd in three cases, and the same happens if I take an odd digit of the first dice (8 answers, 3 from P- students).

5 other students had produced other kinds of texts: "(...) I do not remember the solution"- a P-student; "It should be indifferent but I cannot explain why"; "Half of the numbers of each dice are even" (two answers, 1 from a P-student).

The knowledge at stake concerns the classical definition of probability (as the ratio between the number of favorable cases and the number of all possible cases, provided that they are equally likely), particularly the crucial condition of "equally likely cases". Usual approaches to probability in high school (when required in the programs) do not focus enough on this condition; exercises and examples on most texts concern only equally likely cases (without putting this aspect into evidence).

Individual solutions were written on a sheet of paper and distributed to all students; a discussion followed. *The translation has "smoothed" some students' expressions, due to difficulties of finding equivalent English expressions. Only main steps are reported below. The whole texts in Italian and English will be available for discussion.*

Some supporters of solution B quickly recognized to have considered only one of the two possibilities and abandoned it; but we will see that some doubts about this issue still emerged during the discussion. Then:

1-S1:	I have studied probability at the high school, I have applied the definition of probability as the ratio between the number of favorable cases and the number all possible cases... The possible cases are 11, the even cases are 6, the odd cases are 5, thus the probability of an even sum is 6/11, bigger than 5/11.
2-S2:	but this is not compatible with the precise reasoning of D. (...)
5-S5:	we could make an experiment... to cast the dice a lot of times, one hundred, two hundreds, and see which solution is correct!
6- S2:	we are making a discussion concerning mathematics, we must reach a conclusion by reasoning, not by experiments! Like in geometry, where measurements are not accepted to validate statements! (...)
10- S3:	the dice are concrete! And if we cast two dice a lot of times we can see who is right (even or indifferent).
11- S7:	but we are reasoning in our head, by considering different possibilities and reasoning about them. (...)
13- T:	let us try to see if we can get a shared conclusion by reasoning!
14 -S9:	I would like to come back to solution C. For me it provides a good justification for the "indifferent" hypothesis
15-S10:	For me C is too abstract. I prefer the justification D.
16-S2:	In C do not have the possible cases, we have general, abstract cases... I am convinced that "indifferent" is the correct answer, but (...)
18-S12:	A fits very well what I have studied in the high school: probability of an event is the ratio between the number of favorable cases and the number of all possible cases!
19-T:	Are you sure? Imagine that you win if the sum is equal to 2. Let us consider the following possible cases: sum 2; and sum different from 2.They are two possible cases, the favorable case is the first one, thus

probability of winning is ½. Would you accept to bet an important amount of money on 2?

20-S11: But in this situation it is evident that the two cases are not.... They are not balanced, the case "different from 2" contains many numbers (like 3, 4, 5, 6, ...), we cannot divide one favorable case by those two unbalanced cases! (....)

23-S13: The example provided by the professor is like a counter-example for the use of our definition...

24-S2: but what does it happen in our situation? Perhaps, the sums 2, 3, 4, 5, 6 are not well balanced cases!

25-S6: There is a contradiction: if we follow A, it is more convenient to bet even; if we follow D, it is indifferent to bet even or odd. Thus some of the premises for A or for D are wrong. (...)

28-T: what are the premises? (...)

30-S6: If the conclusions are contradictory, it means that the premises are not compatible among them! We discussed it several times!

31-S11: I agree with you that one of the premises is right, the other is wrong

32-T: Are you sure that one of the two premises is necessarily right?

33-S3: Yes

34-S6: Not, both premises (though different) might be wrong!

35-T: Try to go in depth in the analysis of the premises!

36-S2: the most sure premise is in the solution D: the idea is to consider all the sums... (...)

40-S4: wait! Now I understand: 2 can be the sum in only one case (1+1), while 3 can be the sum in two cases (1+2 and 2+1)... They are not balanced!

41-S1: but I do not agree that 1+2 and 2+1 are different cases... (...)

44-S4: now I understand! A is wrong, because the cases (the sums 2, 3, 4, ...) are not well balanced! *(some voices of consensus)*

45-T: So, what conclusion can we derive from the above considerations?

46-S3: That A is wrong

47-S2: That D is right

48-S6: I agree that A is wrong, but this does not imply that D is right!

49-T: Try to explain why!

50-S6: I am still convinced that it is better to bet even, though I have understood that the motivation A is not right.

51-S3: But if a A is wrong and D is right, then what follows from D is right!

52- S6: We have not yet proved that D is right!

53-S2: D is the opposite of A... A is wrong, thus D is right!

54-S6: No, D is another thing. It is like if I say that Milan is the capital of Italy, and you say that Naples is the capital of Italy: if I prove that Milan is not the capital of Italy, this does not means that Naples is the capital of Italy! (...)

56-T: does D consider all possible cases? (...)

60- S9:	yes, six cases for each number, it makes 36 cases - all possible cases!
61-S2:	I see: we could represent it by a 6x6 table
62-T:	draw it on the blackboard! *(S9 constructs the 6x6 table)*
63-S4:	now I start to understand - not only D is right, but this allows to get the conclusion that it is indifferent to bet even or odd, (...)
65-T:	is it OK for everybody? *(consensus)*
66-T:	what about C? Does it prove that it is the same to bet even or odd? (...)
69-T:	Can we avoid considering specific couples, like in D? (...)
71-T:	are you sure that C ensures that all cases are balanced without considering couples? *(long silence)*
72-S6:	If a dice would have 5 faces...
73-S3:	But a dice has 6 faces! (...)
76-S2:	D implies C, D is a justification for C!
77-S?:	C allows to get the conclusion "indifferent" but is also necessary...
78-S?:	to get D, C is a consequence of D,
79-S?:	thus a longer road to prove that it is indifferent to bet even or odd!
80-T:	Are you sure that C is not an independent way to get the conclusion "indifferent" ?
81-S3:	It is independent, because a 5-faces dice does not exist!
82-S?:	In the reality... But we could imagine it!
83-S4:	C is an independent way to get the conclusion "indifferent" if we can prove in general that the four cases even+odd, odd+even, even+even, odd+odd are balanced...
84-T:	How we could disprove it?
85-S2:	To disprove it, it would be necessary to find a counter-example...
86-T:	Everybody agrees on it? *(silence)*
87-T:	To invalidate a statement is it necessary to find a counter-example?
88-S4:	No, it is sufficient ....
89-T:	Why sufficient and not necessary?
90-S4:	Because to invalidate a statement we can also make a reasoning ... find a contradiction... without finding counter-examples!
91-T:	Let us search for a counter-example. Have you any ideas?
92-S6:	The five-faces dice...
93- S3:	But it does not exist, and I cannot imagine it
94-T:	Is it really important to have dice? What is relevant in the reasoning C? Where digits come from, or...
95-S2:	The fact that half of digits are odd, and half are even!
97-S6:	Thus we could imagine another source of digits...
99- S4:	For instance, digits taken from two boxes.. each box could contain 5 tickets!
100-S2:	Let us see with the table:

(the 5x5 table is constructed; counting results in 13 even sums and 12 odd sums)

101-S6: But then it would be easier to imagine boxes with 3 tickets!

(the 3x3 table is constructed; counting results in 5 even sums and 4 odd sums)

In the economy of the course, if we consider the aim of introducing students into elementary probability theory, the reported discussion can be considered successful not only because crucial aspects of the concept of probability have been focused and clarified, but especially because the subsequent activities put into evidence that the most important "content" knowledge had been learnt by most students. Indeed the individual task "*How to explain to a student that was not there how we got the solution*" was accomplished by 36 students out of 45 with correct and enough precise justifications of the solution "indifferent" (34 students preferred to take the solution D and complete it). More interestingly, after the discussion of some unsatisfactory individual texts another individual task was proposed (the request was to identify in which random situations of a list it was possible to calculate the probability by applying the definition to the given set of cases, and to explain why): 42 students out of 45 were able to identify all the 3 appropriate situations out of the 5 proposed to them, with exhaustive specific justifications for the rejection of the others.

Thus, the "content" aim was achieved. But later I realized (reconsidering the reported discussion) that the potential inherent in some of its "segments" had not been exploited. With the exception of two students (S2 and S6) no trace of the epistemological, logical and meta-mathematical considerations was reported in the individual texts after the discussion. It was like if the attention had shifted from what was required in the task, to the usual presentation of a learned proof (according to the prevailing activity in high school). I must add that during the discussion, the teacher (myself) was aware of the importance of some interventions of the students (this was the reason that induced me to keep the audio-recording till now); what was lacking was not only the time needed to exploit the offered occasions, but also a broad perspective where to situate both "content", and "epistemological", "logical" and "meta-mathematical" aims (elaboration not yet available at that time).

DISCUSSION OF THE DOCUMENT

In this section I will present some tools to frame the reported discussion, and I will use them to analyze the discussion and prepare possible follow-ups.

The culture of theorems

By this synthetic expression I will mean both the knowledge needed to master conjecturing and proving and the capacity to use it, with reference to the construct of theorem (Mariotti & al, 1997) as the triad that consists of: the statement of the theorem; the theory, which the theorem belongs to; and the proof of the theorem, performed within the theory (according to the inference rules and by exploiting the reference knowledge provided by the theory). Knowledge inherent in the culture of theorems concerns the rules of the game of conjecturing and proving. It includes meta-mathematical knowledge about the nature of the acceptable references for the validation of a statement, the role of counter-examples, the logical and textual requirement of a statement and a proof, etc. It includes also more general (logical) rules of arguing. As to the capacity of using knowledge, I will elaborate on it in the following subsections.

This description makes evident the fact that the culture of theorems is part of what we could call "the culture of argumentation". The interest for prospective primary school teachers of the discussion reported in the enclosed document depends on the fact that the development of skills related to argumentation belongs to teachers' professional duties (explicitly stated in the present national programs and national curricula of several states, including Italy). The occasions offered by the document concern in some cases specific requirements of proof, in other cases more general requirements of argumentation; thus prospective teachers can relate reflections on the rules of argumentation to specific issues concerning further mathematics education.

The content of this paper could be rephrased now by saying that it deals with the problem of passing over to students the culture of theorems and of argumentation.

Framing argumentation and proof: Toulmin's and Habermas' models at work

B. Pedemonte in her thesis (2003) and then in Pedemonte (2007) proposed the use of Toulmin's model to study the relationships between argumentation and proof. In Toulmin’s model an argument consists of three elements:

(claim): the statement of the speaker; (*data*): data justifying claim, (*warrant*): the inference rule, which allows data to be connected to the claim.

In any argument, the first step is expressed by a viewpoint (an assertion, an opinion). In Toulmin's terminology the standpoint is called the claim. The second step consists of providing data to support the claim. The warrant provides the justification for using the data as a support for the claim. The warrant, which can be expressed as a principle or a rule, acts as a bridge between the data and the claim. Three other elements that describe an argument can be taken into account: (*backing*) the support of the rule; (*qualifier*) the strength of the argument; (*rebuttal*) the exception to the rule. The force of the warrant would be weakened if there were exceptions to the rule: in that case conditions of exceptions or rebuttal should be inserted. The claim must then be weakened by means of a qualifier. Backing is required if the authority of the warrant is not accepted straight away.

Following Pedemonte (2007) we can apply this model to analyze specific points of the discussion; for instance, in the first intervention the student's claim (it is more convenient to bet even) is related to data (the possible sums) through a warrant (the way of evaluating probability as the ratio between the number of favorable cases and the number of all possible cases) that will be weakened by the remark that another reasoning brings to a conclusion different from the claim.

Habermas (2003, ch.2) distinguishes three interrelated components of rational behaviour: the epistemic component (**epistemic rationality)** inherent in the control of the propositions and their chaining, the teleological component (**teleological**

rationality) inherent in the conscious choice of tools to achieve the goal of the activity, and the communicative component (**communicational rationality)** inherent in the conscious choice of suitable means of communication within a given community). With an eye to Habermas' elaboration, in the discursive practice of proving we can identify (see Boero & al, 2010): an epistemic aspect**,** consisting in the conscious validation of statements according to shared premises and legitimate ways of reasoning (cf. the definition of "theorem" by Mariotti & al., 1997); a teleological aspect, inherent in the problem-solving character of proving, and the conscious choices to be made in order to achieve the aim; and a communicative aspect, consisting in the conscious adhering to rules that ensure both the possibility of communicating steps of reasoning and the conformity of the products (proofs) to standards in a given mathematical culture.

Our adapted Habermas model allows to consider some pieces of the transcript under the perspective of rational behavior: for instance, in the last 10 interventions, once students realize that (C) might not stand by itself as a proof that it is indifferent to bet "even" or "odd" without considering (D), the search for an example in which the four cases of (C) are not equally likely is guided by the teacher and performed according to teleological rationality, in dialectic relationship with epistemic rationality.

The complexity of the discussion can be described rather well if we consider the nature of the claims, data, and warrants used by the teacher and the students during the discussion: we could qualify them with the adjectives "mathematical" (like in the case §1), "meta-mathematical" (like in the case §94), "logical" (like in the case §53 - §54), "epistemological" (like in the case §6).

The integrated use of Habermas' and Toulmin's models proposed in Boero & al. (2010) allows us to analyse the reported discussion according to a broader, unified perspective. The discussion can be considered from two points of view: the point of view of the teacher and the point of view of the students. The teacher's intention is to bring students to realize why the condition of equally likely possible cases is necessary to get a way of evaluating probability that makes sense. Thus his teleological rationality consists of interventions that address the students' attention towards data (produced by the students, or provided by him -see §19) that could weaken or confirm their warrants; while his epistemic rationality allows him to take under control the evolution of the argumentation from the point of view of logical and mathematical validity (see §87), and communicational rationality results in the formulation of interventions that are suitable to reach the students.

From the students' point of view, their teleological rationality is exercised in few occasions and only locally (according to short term aims - see for instance §101) during the discussion, while the path towards the global achievement of the final aim and several steps to approach it are in the hands of the teacher. On the contrary, the students' epistemic rationality is locally at work in an autonomous way in several occasions, both at the mathematical level and at the metamathematical, logical and epistemological level. The task "*How to explain to a student that was not there how* *we got the solution"* fulfils the aim of re-constructing a mathematical, unitary solution of the problem. Did it allow the students to exercise rationality on the global level in all of its components? The lack of reflexive traces in almost all the texts prevents to answer. It is probable that several students wrote down what they were accustomed to do since the secondary school - a well written presentation of the solution of a task, without exercising awareness of the epistemic and teleological requirements of proving. Thus the task does not fulfill the aims of letting the students recompose the rationality of the guided construction of the solution, and of making them aware of the components of the rational behavior at the epistemological, meta-mathematical and logical level. The students' experience in our previous courses allowed some of them (less than 10 out of 45) to raise or deal with epistemological, meta-mathematical and logical issues during the discussion. Those issues remained as concerns for the students who introduced or discussed them, and in most cases were even lost in their individual texts.

The follow-ups

The analysis of the transcript has put into evidence some important elements of the culture of theorems (and, more generally, of the culture of argumentation), which some students have brought to the fore with their interventions. In order that those elements become shared and conscious acquisition for all students, it is necessary to design suitable didactical situations. Epistemological and meta-mathematical concepts cannot be introduced through definitions (specially when young students are concerned). Thus it is necessary to exploit the *occasions* offered by the discussion to approach those concepts through meaningful examples and situations (that will become "reference situations" for them). Various solutions can be worked out.

In a situation like the reported one, the analysis of the main *motive* of the discussion for the teacher (and the students as well) brings us to exclude the introduction of systematic reflections on epistemological and meta-mathematical issues during the discussion. It would mean to distract the students' (and the teacher's) attention from its important "content" goal. Thus it is necessary to design a-posteriori didactical situations based on the use of the transcript of the discussion. The transcript can play the role of a permanent mediation tool, useful for addressing attention to elements that remain available to all students for reflection.

According to experiments performed in similar circumstances (see Douek, 2009 and Boero & al, 2010, second part), I think that a possibility might be to choose the final text by S6 (the most rich on the logical and epistemological ground, even if far from being exhaustive!) and ask students to compare it with another "poor" (but satisfactory from the mathematical point if view) text in order to identify the differences, then ask to identify further aspects of the discussion concerning *similar* (epistemological and logical, according to our terminology) aspects.

Another idea might be to present a complete, reasoned re-construction of the problem (where explicit epistemological, logical and meta-mathematical considerations inform and guide the sequence of steps of reasoning needed to achieve the solution), and ask students to identify if, how and when in the transcript (and eventually in students' individual productions) those considerations emerged.

A third idea might be to analyze the first part of the transcript with students, putting into evidence the relevant logical and meta-mathematical aspects, then ask them to complete the analysis of the transcript according to the same categories.

The three proposals are based on the exploitation of written texts; the first two imply a comparison of at least two texts. This choice provides all students with permanent tools for reflection on the issues at stake.

All the proposals need to be followed by a whole class discussion, orchestrated by the teacher, about (some of the) individual productions. According to the students' maturity and the aims of the activity, during the discussion or after it the teacher might introduce some technical terms and explicit logical and meta-mathematical concepts in order to bring students from an initial, informal awareness of the issues at stake to a more mature take in charge of elements of the culture of theorems (and of the culture of argumentation). More importantly, during the discussion the teacher could drive the students' attention towards a re-construction of the whole elaboration of the solution, in order to make them aware of his intentions and the aims of his interventions. A way to contribute to pass over to the students his rationality.

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