MULTIMODAL DERIVATION AND PROOF IN ALGEBRA

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This paper is a combination of a theoretical paper and a case study. The theoretical purpose is to introduce the concept of multimodal proof into the context of algebra in mathematics education. The case study follows two teacher training students conjecturing and proving results in an environment of visual and physical number patterns. The case study is used both to discuss what multimodal proof can be in this environment and to show some of the potential of such reasoning in the learning of proof.

INTRODUCTION

The interest in visual proof and visual reasoning has been increasing over the last decades. This is the case both inside mathematics and in mathematics education. Several references are given in Bardelle (2009). Barwise and Etchemendy (1994) has developed a computer based proof environment called Hyperproof. Logic students carry out proofs with this program that combines sentential formal logic rules with visual reasoning related to positions and sizes of boxes and pyramids. Barwise and Etchemendy (1996) use the phrase heterogeneous proof for this kind of proofs. This name points to the two distinct kinds of reasoning involved. Oberlander, Monaghan, Cox, Stenning and Tobin (1999) characterize hyperproofs as multimodal, because of the use of both graphical and sentential methods. We think that also other modalities are relevant for mathematical proof. In the context of mathematics education it is possible to involve students in proof like activities which not only uses the sentential and visual modality, but also the tactile, speech and motor action. We introduce a general concept of multimodal proof that intends to cover both formal proofs, visual proofs and proof like student activities. We are not claiming that multimodal proof should be seen as legitimate rigorous mathematical proof. The phrase "visual proof" is widely used even if many mathematicians do not accept such reasoning as part of rigorous proof. In the same way multimodal proof may be seen as a kind of intuitive proof.

This paper analyzes one empirical case study with teacher training students to show what multimodal argumentation and proof can mean. The learning of proof is difficult for students, especially if their background in formal mathematics is week. Proving in the context of visual and physical number patterns gives meaning and makes intuitive thinking available for the students. Seeing the conjecturing and arguing process with number patterns as multimodal proof, makes clearer the potential this kind of activity has for proof learning. This uncovers structure in the students' thinking similar to traditional proof but in other modalities. Those similarities explain why we use the term "proof", and not just multimodal argumentation.

THE CONCEPT OF MULTIMODAL PROOF

In traditional formal proof, the emphasis is on the modality of written symbols and sentential reasoning. A multimodal proof is a generalized proof which beside written symbols and sentential reasoning also can include the visual modality, speech, the tactile and motor action. Gestures are included in the latter. Multimodality means that more than one modality is involved. Barwise and Etchemendy (1996) make a point that visual and non-visual proofs are not separate worlds. Visual proof is, of course, highly dependent on the visual modality, but the mathematician looking at such proofs invokes both language and mathematical concepts in his thoughts. Moreover, proofs not mentioning any picture or diagram have visual aspects. The first argument is that written mathematical symbols by themselves are visual (Rinvold, 2007). Secondly, mathematicians almost always relate intuition to their proofs. This intuition is very often connected with visual ideas, for instance see Sfard (1994).

It is not the intention of this paper to give a precise definition of what a multimodal proof is. This has to be done in forthcoming papers. At the moment we make a first step. In order to increase precision, attention can be restricted to subclasses of the general concept. The concept of proof environment may be useful. The proof environment is the available and allowed resources, objects, operations and modalities. In a sense we have one concept of multimodal proof for each possible proof environment.

A formal proof can be seen as symbols on paper. Also multimodal proofs can be mediated by books, as for instance in Nelsen (1993, 2000). The inclusion of gestures, speech and motor action, means however, that multimodal proof has to be linked to human cognition. More specifically, we are talking about sensuous cognition, Radford (2009). We adapt the semiotic-cultural perspective. "Thinking is considered a sensuous and sign-mediated reflective activity embodied in the corporeality of actions, gestures, and artifacts." (p. XXXVI). In the environment of our case study, gestures, tactility and motor action is considered both to be genuine parts of cognition and multimodal derivation.

A difference between text based proof and multimodal proof is the non-linearity of the latter. In the "proofs without words" of Nelsen (1993, 2000), some proofs are given holistically by diagrams with limited guidance to the reader. Traditional proofs are given step by step. Numbering of formulas and names of applied theorems are ways to guide the reader. When the carriers of information are physical, gestures is an important way of directing the attention. More specifically, pointing gestures is used for this. The research literature uses the name *deictic gestures*. According to Sabena (2008) "*deictic gestures*: indicate objects, events, or locations in the concrete world" (p. 21). Some words in written language or oral speech have a similar function. Radford (2009) gives the examples "top" and "bottom" (p. XLI). These kinds of words are called “spatial deictics”, (p. XLI). Other pointing phrases are “this one” and “that one”. Both linguistic and non-linguistic deictics are central to multimodal deduction. In classical proof we use mathematical deictics as “equation 25” or “theorem 4”, which means that neither this kind of deduction is completely linear.

ALGEBRAIC THINKING IN MATHEMATICS EDUCATION

In mathematics education the multimodal perspective has been used to analyze student thinking, especially in algebra and functions. Radford, Edwards and Arzarello (2009) introduce what they called “construction of mathematical meaning from the perspective of multimodality”:

> ... taking into account the range of cognitive, physical, and perceptual resources that people utilize when working with mathematical ideas. These resources of modalities include both oral and written symbolic communication as well as drawing, gesture, the manipulation of physical and electronic artefacts, and various kinds of bodily motion. (Radford, Edwards and Arzarello 2009, p. 91)

Our use of the word multimodal is inspired by and compatible with the perspective of those researchers. Radford (2009) have used a multimodal approach to the introduction of students to the elementary parts of algebra. We share the context of visual number patterns in algebra with Radford, but our focus on proof is new.

Radford classifies algebraic thinking into three forms, factual, contextual and symbolic. Following Radford, contextual thinking acknowledges a general figure through the concept of figure number, but “still supposes a spatially situated relationship between the individual and the object of knowledge...” (p. 9). The students’ reasoning depends on the particular context and perspective. When the students start to use symbols for variables, but still think contextually, Radford observed a phenomenon which he called iconic formulas or formulas as narratives. Formulas are not simplified, but divided into parts. Each part tells a story about the corresponding part of the pattern described by the formula. We identified both the use of contextual thinking and iconic formulas by the students in our case study. Contextual algebraic thinking is not necessarily a low level of thinking, but the students also need to master the symbolic form in order to use multimodal thinking in a mature way.

LEGITIMACY OF MULTIMODAL PROOF

In pure mathematics, visual proofs are still not widely accepted as legitimate part of rigorous proof. However, arguments have been put forward that may change this in the future. Barwise and Etchemendy (1996) have done work in mathematical logic which argue for the legitimacy of some kinds of visual reasoning. In philosophy of mathematics Brown (2008) argues for the possible validity of pictures in proofs. Jamnik, Bundy and Green (1997) and other studies in the field of automated reasoning has also indirectly provided good arguments for the case of visual proof. They have shown that typical diagrammatic proofs can be formalized with recursive ω-logic and automated. This is an argument for the legitimacy of a wide class of visual proofs. These proofs are based on generic reasoning. In such proof the general is proved through special cases. One or two instances are used to describe how to carry out the proof for any instance. Recursive ω-logic is an alternative to the classical induction approach, but equally valid. In ω-logic a formula is proved if you have a proof for each instance of the formula. A recursive ω-proof also has an algorithm to find the proof of each given instance. The use of physical figures and number patterns in our case study are confined to generic reasoning. The validity of multimodal reasoning in these kinds of environments thus rests on the correctness of generic arguments.

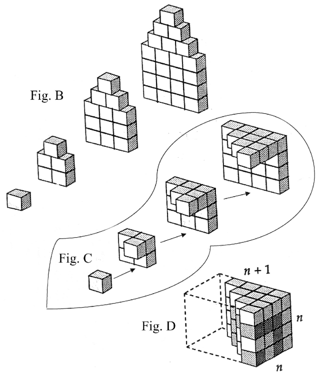
Conditions for legitimate formal reasoning are well developed. Similar analyses of visual reasoning are still in an early phase. The work of Barwise and Etchemendy (1996) is relevant also for multimodal proof, but the possible legitimacy of multimodal proof is not yet analysed in depth. An idea of Barwise and Etchemendy (1996) useful for such analysis is the concept of information. "Valid deductive inference is often described as the extraction of making explicit of information that is only implicit in information already obtained." (p. 4). Mathematical formulas and written text contain information. Classical proofs show how the information given by a theorem is already implicit in axioms and established mathematical results. Also pictures, diagrams and physical patterns contain information. A map, for instance, contain lots of information. If you know the use of map and compass, you can deduce how to get from A to B. As in classical proof, some deductions are valid and others not. You do not get from A to B if you do not use the map and compass in legitimate ways.

METHODOLOGY

The paper uses one single successful case study with teacher training students to give examples and to discuss what multimodal argumentation and proving can mean. The students are in a problem solving process leading to derivation of formulas. We do not see a straight line of reasoning, but also some mistakes and dead ends. As such, the students are proving, not presenting a proof. The data gives examples of multimodal proving, but much of the argumentation could with some refinement be part of a multimodal proof. We thus see continuity between proving and multimodal proof. Within a traditional proof paradigm, more radical changes are needed to go from the students' argumentation to proof. However, we do not claim that this kind of continuity between argumentation and proof always is the case within the multimodal paradigm.

In the spring of 2009 five pairs of teacher training students took part in a study conducted at Hedmark University College and NLA University College. All of the students were following a course in number theory at one of the two colleges. The pairs were asked to investigate number patterns represented by visual and physical figures. Their work is immediately followed by an interview by one or both researchers. Both the student investigations and the interviews were videotaped. One episode from one of the pairs is chosen for analysis. The selected pair clearly was the most successful. We observed a complex interplay between the students and between visual diagrams, physical figures, speech, gestures and formulas.

THE START OF THE CASE STUDY



The students Erik and Jon are given a problem sheet with two equivalent visual number patterns B and C and asked to find out as much as possible about the sequence. Also included is a figure D. The latter figure results when the four figures in C are sliding together. In front of the students are physical versions of C and D built by plastic cubes. More cubes are available so that they may build their own figures. Each student also has a personal note sheet. Before we come into the story, the students have spent about 20 minutes. Among other things, they have derived an explicit formula for the sequence in B by decomposing each figure into a square and a triangle on top of it. They also observed that B and C give the same sequence.

THE FORMULA TO BE PROVED

We will follow the derivation of a recursive formula for the sequence in B. The students were not asked to prove such a formula. They discovered and proved the formula themselves. In the problem solving process the given recursive formula was the end product. They first wrote the square part of the formula and then added the triangle part. As can be seen, F_{n-1} was written at a later stage.

$$F_n = F_{n-1} + \underbrace{2(n-1)+1}_{\square} + \underbrace{n-1}_{\triangle}$$

F_n is referring to the n'th figural number from the left in B or C. The formula is a combination of standard mathematical symbols, curly brackets and drawn figures. The latter are signifying the square and triangle part of the formula. The formula to be proved is not part of standard notation. The drawn square and triangle link the formula to the multimodal context of the activity. Following Radford (2009), the formula could be seen as iconic or as telling a narrative. This means that the formula mirrors the division of each figure into a square part and a triangle part.

In the following analysis, we use the students' argumentation to show what a multimodal proof can be. Deictics has a central role. We also look for generality in the students' thinking. One example is the use of indices. It is not mature, but nonetheless they think generally through them.

THE BEGINNING OF THE DERIVATION

The start of the derivation is an oral description of the task:

105 Jon: We are going to find a formula to find the next one when we know this one.

Descriptions, definitions and reformulations often are the start of direct proof. No deictic gestures are used in 105. This may be interpreted as generality in the students' reasoning. Then they enter a journey outside the main path, but after a while Jon reformulates 105 with gestures and symbols on his note sheet. He has already written "$F_n = F_{n-1} +$ ".

139 Jon: To find F_n [pointing at F_n on his note sheet] it's the last one [circular pointing gesture around F_{n-1}] plus something more [points to the right of +].

Both deictic gestures and deictic speech turn up already at this juncture when standard mathematical symbols appear. The last deictic gesture is abstract pointing. According to Sabena (2008), abstract pointing is "when there is no actual physical pointed object, rather the pointed empty space houses an introduced reference, ..." (p. 22). Then Jon introduces the visual B-pattern on the problem sheet. The second and third figure in this pattern are denoted $F_2(B)$ and $F_3(B)$.

141 Jon: It becomes this [circular pointing gesture around $F_2(B)$] plus something more [points in direction of $F_3(B)$].

The deictics shows the connection when 141 is derived from 139. Note the striking similarity between gestures linked to corresponding terms, for instance the circular pointing gestures used both for F_{n-1} and $F_2(B)$.

THE LEMMA

At this juncture both students start to manipulate the physical figures. Jon puts F_2 and F_3 together and then takes one of these figures in each of his hands. After a short period of silence Jon expands what he wrote in 139 to this formula:

146 Jon: $F_n = F_{n-1} + (K_n - K_{n-1}) + (T_n - T_{n-1})$

K_n and T_n are referring to the n'th square numbers ('Kvadrattall' in Norwegian) and triangular numbers.

147 Jon: This one is nice [smiling]! But, is it correct?

The question about correctness means that 146 at this stage is a conjecture or a lemma to be proved. The lemma splits the problem of 105 into two parts. Jon's smile and description of formula 146 as nice may be an indication that this is the kind of answer which is expected in mathematics. But, we will see that the formula is used more as a vehicle of thought than in the standard way. Careful readers may have noticed that the indices for the triangular number part are wrong. The students, however, do not comment on this “mistake”, but get it right when $T_n - T_{n-1}$ is replaced by n – 1. The indices n and n – 1 at this stage seem to have the operational meaning “the next one” and “this one”. One argument for this is the complete similarity between Jon’s derivations of the triangle part and the square part of 146. Because of this similarity we only show Jon’s derivation of the square number part:

149 Jon: From this one [pointing to physical F_2] to this [pointing to F_3], this square number is added [lay his fingers down on F_3] minus the square number we had earlier [makes a circular pointing movement around the bottom of F_2]. K-n minus K-n minus one.

In fact, Jon uses 141, but he has replaced the visual B-patterns with the physical figures. 149 is the first time physical figures are explicitly used in the derivation. Deictic gestures now have a vital place in the derivation to show which part of the figures which is in focus. Note how the physical figures give more clear and precise gesture possibilities compared with the visual B- and C-figures.

THE FINAL PART OF THE DERIVATION

Now we come to a striking example of multimodal proving involving physical objects and gestures. The lemma (146) is proved, and the students return to the derivation of the recursive formula. They are going to replace each part of the lemma with a simplification.

152 Jon: [Laughs] I doubt the formula should look like this.

153 Jon: It isn’t very nice, but [smiles and laughs] [silence]

154 Jon: Yes, gets right, yes, but probably has to be rewritten in one way or another.

At this moment Jon realizes that the lemma can be simplified and starts to think how. We will follow the simplification of the square number part. The reasoning for triangle part is similar. Jon has changed his mind and does not find the formula “nice” anymore. Jon’s question is trigging Erik to make a contribution. He takes F_3 and puts it on top of F_4:

159 Erik: What is added then? It is the sides, the sides of the first quadrilateral number plus one. [He moves his finger along the side while talking.]

160 Erik: Like quadrilateral numbers...

161 Jon: n minus one then

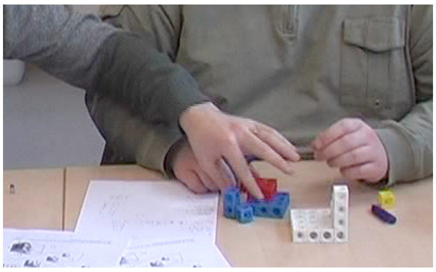
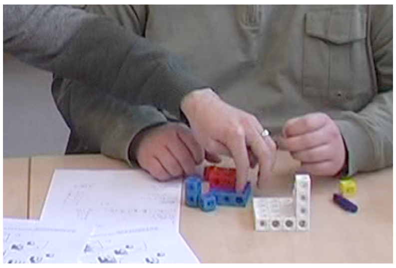
162 Erik: Two times minus one plus one. [He writes a formula on his note sheet.]

$$F_n = F_{n-1} + \underbrace{2(n-1)+1}$$

163 Erik: The corner is included. [He points to the corner between the mentioned sides.]

Erik locates the square numbers as the bottom part of the physical figures. He physically compares the bottom squares of F_4 and F_3. The way the figures are relatively placed allows him to show one of the added sides with a sliding finger gesture. The plus one is explained by pointing to the corner below the triangular number part of F_4.

This explanation by Erik is an example of reasoning which can be refined. His idea is good, but the triangular number on top of the bottom square obscures the reasoning. In fact, Jon does not understand, and Erik follows up by an improvement. Erik takes off the triangular parts of F_2 and F_3 and places K_2 on top of K_3.

Then he shows a sliding finger movement along each of the two sides and points with a finger to the corner. Now it is obvious to Jon what is going on.

THE GENERALITY OF THE REASONING

The interview indicated that the concept of figure number was a safe ground which resolved potential dangers in general reasoning. Again, this corresponds to the contextual form of reasoning in the classification of Radford (2009).

341 Jon: We noticed by looking at the drawing that the square number equals the figure number.

342 Researcher: Yes.

343 Jon: And the triangle number is one less.

Except for the non-standard use of indices in the lemma, they always related figure numbers correctly to symbolic variables. This indicates that the students were aware of the generality of their reasoning. Their switching between different values of n when relating to physical figures strengthens that conclusion. In "The final part of the derivation" Jon uses n = 3 and Erik n = 4, but they do not comment on the difference. Jon and Erik behave as if they are talking about the same thing. A few minutes later Erik repeats his sliding finger gesture argument with n = 3, also with no comments. Jon was convinced that the reasoning done for a particular value of n could be done for all other values as well. They have tested the formula in the theorem for n = 4, but not for n = 5. When the researcher asked for reasons to trust the formula, Jon repeated the derivation from the third and fourth physical figures. Then the following dialog ends the interview relating to the recursive formula:

428 Jon: You really see that it's logical both from the figures and from the plastic cubes.

429 Researcher: Yes, you could have built something similar if we considered number five from four, for instance?

430 Jon: Yes! [Looking convinced]

CONCLUSION AND QUESTIONS FOR RESEARCH

Traditional proof has been thought to consist only of sentential reasoning. Visual proof in an active way uses visual information. Multimodal proof can also include the use of physical objects, the tactile, gesture and other kinds of motor action. Especially the latter modalities transcend the traditional concept of proof.

The analysis of data has shown that quite advanced multimodal proving is possible for students even if their form of algebraic thinking partly is contextual and only to some extent symbolic. There is much structure in their proving even if they are not trained in formal proof methods. A possible explanation is the resources of intuitive thinking which is opened by this approach. A conclusion is that this kind of proof activities has a potential in proof learning. The data analysis has also shown the important role of deictics in multimodal reasoning.

The concepts of multimodal proof and derivation need further clarification and development in order to discern valid proof from other kind of activity and presentation. We think that the concepts of proof environment and extraction of information will be helpful in this. Further research is needed both to find conditions for valid reasoning and to investigate the role of deictics in multimodal proof. The latter may also be a key to better understanding of visual proof. In mathematics education more empirical studies are necessary, combined with development of better design and teaching approaches. A question of research is how to support the development of students reasoning to include the symbolic form of thinking.

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