STUDENTS' UNDERSTANDING OF PROOF BY CONTRADICTION

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Two hundreds and two students of 17~20 years old were surveyed on their understanding of proof by contradiction. Five abilities were identified for interpreting their understanding. A two-streamed model of understanding proof by contradiction was constructed statistically. To analyze the negating of a statement with quantifier 'only have one', interviews were conducted to reveal the relationship among the language used, Chinese or English, in their thinking process and their logical judgment.

INTRODUCTION

Understanding proof by contradiction shall mean to have both the procedural and conceptual knowledge of proof by contradiction. The procedure knowledge is: negating the conclusion q, and then inferring a mathematical fact or assertion that is contradicted to p. The conceptual knowledge is: "if ~q then ~p" implies "if p then q". This step, which is the principle of proof by contradiction, is based on the law of contrapositive. Historically, proof by contradiction is a method necessary in constructing mathematics systems. It was realized by today's mathematicians that mathematics systems might not be constructed if proof by contradiction has not been used in mathematics.

Young children may already have experiences informally of using reasoning with contradiction in their playing. It was found that 7~8 years old reasoned using contradiction in game playing and in checking conjectures (Reid & Dobbin, 1998.) However, the following three issues about the learning difficulties of this indirect proof method were notified in the related studies. These issues have motivated us to carry out this study.

The first is "When to use proof by contradiction?" Analyzed the interviews with six mathematicians about when they would think of using proof by contradiction, two criterions were mentioned: (1) the given conditions are not able or not easy to be manipulated; (2) The negation of conclusion reveal an obvious representation within a familiar system. These criterions are trivial to mathematicians, but are not familiar to senior high school students. For proving the irrationality of $\sqrt{2}$ is just their first experience of using proof by contradiction. Barnard and Tall (1997) studied the difficulties experienced by students of 16~19 years old on proving $\sqrt{2}$ is irrational. They highlighted six themes to show the difficulties. The initial one is the overall notion of proof by contradiction. This overall notion might include the explanations of 'what is' and 'when to use' proof by contradiction. Reid and Dobbin(1998) suggested that the difficulties students have with standard proof by contradiction in mathematics may arise from issues of the need from which their reasoning arises. When a student is asked to read the proof of $\sqrt{2}$ is irrational, what needs drives that proving? They argued that it is very rare that a need to verify comes into play. To feel a need to verify one must be uncertain of the result. In the case of $\sqrt{2}$, it is unlikely that there is any uncertainty at all.

Ability of Negating a Statement

Procedurally, negating a statement is the first task on processing proof by contradiction. Mathematical statements very often contain certain quantifiers, such as: all, only one, some of, don't exist, etc. Abilities of negating a statements with/without quantifier shall be one focus of this study.

Abilities of Reasoning Contrapositively

Zepp; Monin and Lei(1987) developed a test using implicational and disjunctive sentences for studying common logical errors in English and Chinese. They found that 54% of their university subjects were able to use the law of contrapositive in their reasoning on content-free tasks. Regarding to this facility, they argued that if the learning of logic principles depends on one's experience, it is possible that different logical principles may be learned for different situations. The logic in the mathematics classroom may be different from that in daily life environment. Thus, this study intended to develop items in both mathematical and daily life contexts for the investigation. In summary, this study aims for investigating abilities of negating a statement and their thinking process, recognizing the law of contrapositive, recognizing the procedure of proof by contradiction, and for constructing a model of understanding proof by contradiction.

METHODOLOGY

This study were conducted as following: (1) paper and pencil questionnaire; (2) individual and focused groups interviews; (3) field testing: In order to find a suitable model, items were analyzed. Percentile for each item was used to analyze the performance level of the samples. Factor analysis was used to analyze the abilities needed in the process of doing a proof by contradiction method. (4) proposing the model of proof by contradiction; (5) teaching experiment: After the model was proposed, a three hours teaching experiment were conducted to verify the model (Lin, F. & Cheng, Y. 1997. pp.557-591). The teaching experiment is not described in detail in this paper.

Item Development

Items in the questionnaire are grouped into three categories. Each category includes items in the context of daily life and the context of formal mathematics. Category 1: negating a statement. This is considered as a prior ability for proof by contradiction. Eight items belong to this category with different contexts and different quantifiers. Category 2: recognizing the procedure of proof by contradiction. Two items belong to this category. One asked students to recognize the procedure of proof by contradiction, the other asked students to state the procedure. Category 3: recognizing the law of contrapositive. Four items belong to this category, three are in the context of daily life, and one is in the pure mathematics context.

Interviews

Two kinds of interviews were conducted in this study. Individual interviews were conducted for testing the clarity of the items and exploring students' understanding. Interviews with focused groups were conducted to understand the thinking process of students and teachers related to negating a statement in daily life. During the interviewing process, we have observed that in the thinking process the language obstacles, Chinese or English, affected their performance. In order to clarify the relationship, we conducted

some interviews in the ordinary class hours. Statement with the quantifier "only one" was used to interview a class of university students and a class of junior high school mathematics teachers in a summer program, and graduate students.

Subjects

One hundred and forty 11th grade students and sixty-two college students major in mathematics or mathematics education participated in the questionnaire survey. According to the college entrance examination data (Lin, 1991), the distribution of these students' scores was similar to the distribution of the scores of the whole population of senior high school graduates in Taiwan. Seventy-one samples, mathematics majors or junior high school in-service teachers, participate in the group interviews.

RESULTS

Negating a Statement

Table 1 gives the correct frequency of two items and the classification of each question according to the quantifier terms and the context.

Table 1. Facilities on Items of Negating a Statement

Item No.	1-1	4-1	1-4	4-4	1-3	4-2	1-2	4-3
Quantifier	none	none	some	some	all	all	only one	only one
Context	Real	Pure	Real	Pure	Real	Pure	Real	Pure
Correct	93.0%	84.6%	82.1%	73.2%	49.5%	53.4%	19.8%	16.8%

The types of the quantifier affect the performances. About 90% were able to negate statements without quantifier; On the other hand, about 80% were not able to negate the quantifier "only one", which is the hardest question in this test. The reason for this will be elaborated later. The followings are some statements with their most frequent errors we found in this study. For example, 40% students negated the statement "all people are my friends" as "no one is my friend", and 16% students thought the statement "all three angle of the triangle ABC are acute" should be negated as "no angle of Triangle ABC is acute". There are 55% students thought the negation of the statement "Engle has only one brother" was "Engle has more than one brothers", and 54% students thought the negation of the statement "the graph of function f(x) intersected x-axis at only one point" was "the graph of the function f(x) intersected x-axis at more than one points".

The effect of context is revealed when the items are easier. For the statements with the quantifier "some" and the statements without the quantifier, the correct frequency of items in the daily life context were about 10% higher than items in pure mathematics context. However, no difference was found for harder items, such as the statements with the quantifier "all" and "only one".

Recognizing and Stating the Procedure

Procedural knowledge of proof by contradiction includes two steps: negating the conclusion, and inferring a result that is contradicted to the assumption or a known fact. Table 2 gives students' response to the related items.

Table 2. Facilities on items of recognizing the procedure

Response type	Recognized proof by contradiction	Given correct explanation	Correctly stated the procedure

	(item 6-3)	(item 6-3)	(item 8)
Correct frequency	83.7%	36.6%	53.9%

About half of the students were able to write the procedure of proof by contradiction. Although 83.7% students could recognize a proof by contradiction, but only 36.6% students were able to give a correct explanation. There is about 20% students who were able to state the procedure but not able to apply it when it was needed. In addition, there were 6.9% students made the mistakes of writing the procedure of proof by method of exhaustion instead of proof by contradiction.

Recognizing and Using the Law of Contrapositive

The conceptual knowledge of proof by contradiction is the law of contrapositive. Students need to recognize that "IF P THEN Q" and its contrapositive "IF ~Q THEN~P" are equivalent. They need also to realize that "IF ~P THEN ~Q" and "IF QTHEN P" are not equivalent.

Table 3. Facilities of Items on recognizing the law of contrapositive

Item No	2	3-a	5-a	3-c	5-c	3-b	5-b	3	5	7
Context	Real	Real	Pure	Real	Pure	Real	Pure	Real	Pure	Real
Types of statement	~p=>~q	~p=>~q	~p=>~q	~q=>~p	~q=>~p	q=>p	q=>p			
Correct frequency	58.4%	49.5%	66.3%	53.5%	48.5%	73.3%	65.8%	24.7%	30.1%	23.7%

Table 3 gives the correct frequency of the four items and the classification of each question according to the type of the statement and the context. More than 70% did not have the conceptual understanding of proof by contradiction. They could not correctly recognize that "If P then Q" is only equivalent to "If ~Q then ~P", but not equivalent to "If P then Q" or "If ~Q then ~P". About 49% thought that each pair of the statements in the items was not the same. Only about half students had the necessary conceptual knowledge, thinking that "If P then Q" is equivalent to "If ~Q then ~P". This result is similar to the result in Zepp, Monin, & Lei (1987). Students with the misconception that "If P then Q" are the same as "If Q then P" is fewer than the above misconception. However, there are still 30% had this misconception. Table 3 also shows that reasoning contrapositively in the context of daily life was a little harder than in the context of pure mathematics.

Language, Thinking, and Logical Reasoning: -- example of negating "only one"

According to Nakamura's study about the thinking of Eastern species (__, ___ 1992), it was found that Chinese language emphasizes on individual, practical objects, and believe in the sense and intuition, but lacking the understanding of the general rules. Therefore, in Chinese language usually the facts were stated, and less attention was paid on the logical reasoning method and skills. On the other hand, western mathematics curriculum emphasizes on the logical reasoning and formal proofs. This difference might cause the obstacles in the thinking process for Chinese students in mathematics learning.

Analyzing students' responses to negating the term "only one", it was found that less than 20% were able to negate it. More than 70% used the term "none" or "more than one" to

negate “only one”. In order to understand this more deeply, this study investigates the relationship between the Chinese language and the logical thinking. The sentence “I have only one brother” was used in the classroom interview. We found that the thinking models shows varieties affected by the words leading thinking or the thinking leading words. For those belong to “Words leading thinking”, they negated the statements by way of speaking or reading, and then translated the conclusion back to the spoken or read language. In this way, the concept of negating is first negating the Chinese words, and then explaining the meaning of the negated words in English or Chinese way. They might be thinking in English. For example, the negation of “I have” is “I don’t have”. So, the statement was negated as “I don’t have one brother”, which is “I have no brother”.Or they might be thinking in Chinese. For example, in Chinese, the words for “have only one” were in the order “only-have-one”. First added “not” to the statement with the term “only-have-one” was negated into “not-only-have-one”. However, the words “not-only” in Chinese means “more than one”, so the conclusion followed as “I have more than one brother”. For those belong to “Thinking leading words”, they negated according to the semantic of the term “have only one”, and then followed the meaning of the negated statement to form the conclusion. They might think in semantic way. In Chinese the semantic meaning of “have only” is “few”. Therefore, negating “have only” would follow as “more” and getting the conclusion “I have more than one brother”. Or they might think in Pseudo-semantic way. For example: (1) “have only one” means “have”, so the negation is “not have”. Students who followed this type of thinking got the conclusion “I have no brother”; (2) Under the assumption of “have”, some students negated the amount “one” and got the conclusion “I have more than one brother”; (3) “have only one” means “have just one”. Therefore, the conclusion is “I have no brother or have more than one brother”. Some student thought that “have only one” is related to a logical term. The Chinese term “have only one” is “exactly one”. So the conclusion is “I have no brother or I have more than two brothers”.

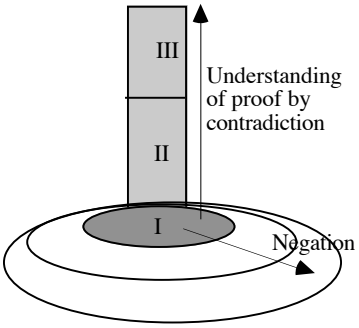
LEARNING FACTORS AND UNDERSTANDING MODEL

SAS pro factor was used to conduct the factor analysis. Only those factors with egenvalues higher than 1 were considered. It was found that there were five major factors that affect the understanding of proof by contradiction. The characteristics of each factors were analyzed and named them accordingly. **F1**: able to negate statements with simplest quantifier. Four items assessed the ability of “negating the statements without quantifiers or with the quantifier some”, in pure mathematics or real world situation. **F2**: able to recognize the law of contrapositive. Four items are related to “recognizing the law of contrapositive”, in pure mathematics or real world situation. Students who gave correct answers were not only able to recognize the equivalence relation of “p=>q” and “~q=>~p”, they were also able to recognize that “p=>q” were neither equivalent to “q=>p” nor to “~p=>~q”. **F3**: able to negate the statements with the quantifier ‘ only one’. Two items are related to “negating statement with quantifier only one”, in pure mathematics or real world situation. Students who correctly answered theses items 1-2 and 4-3 could overcome the obstacles of Chinese language and correctly negate the statements with the quantifier “exist only one”. **F4**: able to describe the procedural knowledge. Two items are related to “recognizing and stating the procedural knowledge

of proof by contradiction". **F5**: able to negate the statements with the quantifier "all". Two items are related to this factor.

In order to establish the development model of understanding proof by contradiction, the method used by the research programe "Concepts in Secondary Mathematics and Science"(CSMS) is adopted (Hart(ed), 1998.) According to the item difficulties, items in this study can be classified into five groups, each corresponding to the five factors discussed in previous section. It was found that items with the item difficulty value higher than 70% were those items related to factor F1; items with the difficulty between 30% and 20% were related to factor F2; items with the difficulty below 20% were related to factor F3. Those items between 55%and 35% can be grouped into two according to the item characteristics and were related to factor F4 and F5 respectively.

This study combines the two dimensions M1 and M2 as a model of the development model of understanding proof by contradiction. Following figure gives a simple description of the model. M1 describes the development of the knowledge of proof by contradiction. F1-F4-F2 describes three levels of M1 from the bottom to the top. M2, which describes the development of the ability of negating statement, is considered as the basic ability of proof by contradiction. F1-F5-F3 described three levels of the abilities of M2.



CONCLUSION

From the result of our study, it might be reasonable to make conclusion about how students learn proof by contradiction. In summary, according to the field test, interview results, and teaching experiments, this study proposed a development model of the understanding of proof by contradiction. The first step of proof by contradiction is to negate the conclusion. After a student is able to negate a basic statement, he/she can begin to learn the procedure knowledge of proof by contradiction. However, only until a student understands the law of contrapositive, he/she will know why the procedure is finished. The ability of negating a statement might be developed unrelated to the understanding of the procedural knowledge of proof by contradiction. In order to verify our conjecture, a teaching experiment has conducted to check the understanding model of proof by contradiction. The result of that experiment shows that it is possible to help student to understand the method and apply it through the real world situations.

This study also found that the difficulty levels of students' negating a statement can be ordered decreasingly as negating statements without quantifier, negating "some", negating "all", and negating "only one". Besides, according to the result of the

interviews, Chinese language might cause addition difficulty in negating statements with some quantifiers. In Chinese language, students negated statements either according to the words or according to the thought. It was found that students might have mistakes in negating a statement because the universal set (whole set) they used were different from the universal set usually used in mathematic. Since this study did not investigate the negating of "or" and "and", it is unable to know whether the negating of statements with these two terms will form another level or include in one of the levels found in this study.

According to the field test, most the students in this study recognized the procedural of proof by contradiction, but only half of them can describe the process of proof by contradiction. Among those who described proof by contradiction method correctly, about one fifth of them could not apply it. More instruction design can be investigate based on this model to help students in understanding proof by contradiction and applying it.

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