

# EMPIRICAL GENERALISATION AS AN INADEQUATE COGNITIVE SCAFFOLD TO THEORETICAL GENERALISATION OF A MORE COMPLEX CONCEPT<sup>1</sup>

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*The impact of prior learning on new learning is highlighted by the case of Dean, a Year 8 student who developed his own method to find the sum of the interior angles of a polygon without knowing why his method worked. Enriched transcripts and visual displays of the cognitive, social (Dreyfus, Hershkowitz, & Schwarz, 2001) and affective elements (Williams, 2002) of Dean's interrupted abstraction process informed the identification of factors that inhibited Dean's constructing process. It was found Dean possessed an empirical, not theoretical, generalization (Davydov, 1990) about sums of interior angles of triangles that was an inadequate cognitive artifact for constructing the new more complex theoretical generalization. The study suggests use of tasks designed with the opportunity develop assumed knowledge in conjunction with new concepts.*

This case forms part of a broader study of factors that promote or inhibit the process of student-initiated and student-directed abstraction of mathematical concepts, without mathematical input from an external source (like the teacher, text-book, or students external to the group) during the abstraction process. Dean's case illustrates an interesting phenomena; his inability to utilize the procedure specified and demonstrated by the teacher appeared to trigger his development of an alternative more complex strategy. Whilst the class as a whole developed the empirical pattern 'you add a hundred and eighty degrees each time', Dean drew upon prior knowledge that he considered to be part of a different topic 'angles in triangles add to a hundred and eighty degrees' to develop his new method 'you add another hundred and eighty degrees for each triangle in the polygon'. Dean was unsure his method was correct (even though he had checked it with specific examples) because he had been unable to 'see' the mathematical essence behind the prior knowledge he utilized; Dean did not know what angles were or where they were positioned in triangles and polygons. Dean had developed an 'empirical generalization' using a less complex empirical generalisation (Davydov, 1990) to scaffold his thinking. Dean did not develop a theoretical generalization (Davydov, 1990) or abstract a new mathematical insight (Dreyfus, et al., 2001). This report examines why a student who had demonstrated the capacity to recognize the relevance of mathematical ideas, that he considered external to the lesson focus, did not also gain mathematical insight into the method he developed.

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## LITERATURE REVIEW AND THEORETICAL FRAMEWORK

Student-initiated and student-directed abstraction (discovered complexity) has been found to be associated with high positive affect and optimization of learning conditions (Barnes, 2000; Csikszentmihalyi & Csikszentmihalyi, 1992; Williams, 2002). Discovered complexity occurs when students spontaneously formulate a question related to a newly found mathematical complexity and work with unfamiliar mathematical ideas to explore this further (Williams, 2002). Discovered complexity is a subset of 'abstraction'; an activity of vertical reorganisation of 'previously constructed mathematical knowledge into a new structure' (Dreyfus, et al., 2001, p. 377). 'Vertical' refers to a new mathematical structure as opposed to a strengthened connection between a mathematical structure and a context ('horizontal'). Dreyfus, et al. (2001) identified observable cognitive elements of the process of abstraction: (a) 'recognising'—seeing a previously known mathematical structure within a new context or realising a previously known mathematical structure fits a new context; (b) 'building-with'—using a combination of previously generated abstracted entities in a new context; and (c) 'constructing'—using assembled resources to vertically reorganise a mathematical structure. In the present study, to facilitate examination of the inhibited process of abstraction, 'building-with' is taken to include use of a rule where the mathematical essence behind the rule is not known; conceptual ideas that enable the justification of a rule or pattern are not present.

The six categories of dialectic social interaction (control, elaboration, explanation, query, agreement and attention) (Dreyfus, et al., 2001) can assist in determining whether ideas are student-initiated and/or student-directed. For example, a student-directed interaction that was not student-initiated would be controlled initially by an external source but once the process of abstraction had commenced, all elaboration, explanation, and agreement would emanate from an internal source. The group would decide whether to attend to any query or attention from an external source. To reduce the difficulties associated with identifying cognitive artifacts assembled by students for use during the process of abstraction, post-lesson video-stimulated reconstructive student interviews can be used to 'make visible' additional cognitive activity (Clarke, 2001). Nisbett and Wilson (1977) have shown people can produce accurate reports of their own cognitive activity if salient stimuli (like video-stimulation) are provided.

Williams (2002) used Csikszentmihalyi and Csikszentmihalyi's (1992) concept of flow to develop indicators of a student's affective state. The state of flow exemplifies the enhanced quality of the learning experience that can occur where task involvement is associated with a high level of positive affect. The indicators of task involvement are: eyes on the task; pens on the task page and/or bodies leaning in towards the task; and, participating in the interaction. A more intense task involvement can be inferred where these previous indicators occur in conjunction with the following indicators: lack of awareness of the world around; building on each other's ideas (latching comments); and exclamations of pleasure. More detail about flow, body language and positive affect can be found in Williams (2002).

In this study, the cognitive and social elements of the process of abstraction are examined in conjunction with the affective indicators 'visible' on the video and 'audible' in the interview to answer the question: Why was the process of abstraction inhibited for Dean?

## RESEARCH DESIGN

The Year 8 lesson studied was the 12<sup>th</sup> lesson in a sequence of 16 lessons in a government school in a lower middle-class area in Australia. The teacher was seen by his school community to display 'good teaching practice'. Three video cameras operated simultaneously in the classroom to capture the actions of the class as a whole, the teacher, and a pair of focus students. Data included videotape of the lessons, post-lesson video-stimulated student interviews, and photocopies of student work and lesson tasks (Clarke, 2001). The video-stimulated interviews were intended to reconstruct the learners' perspective. A student was given the remote control to a mixed video image with the focus students at center screen and the teacher as an insert in the corner. The student was asked to identify and discuss the parts of the lesson that were important to that student.

## RESULTS AND ANALYSIS

The aims and outcomes of Lesson 12 for Dean and for class members in general are now described. The teacher intended students to learn the algebraic rule for finding the sum of the interior angles of a polygon. Class members used a table of student-generated results to find the rule 'add a hundred and eighty degrees each time'. Dean did not attend to the method the class developed; he developed his own method 'add a hundred and eighty degrees for each triangle'. The teacher then used student pattern recognition to formulate ' $(n-2) \times 180^\circ$ ' (where  $n$  is the number of sides of the polygon). Dean demonstrated some proficiency in applying this rule. The evidence now reported is drawn predominantly from: (a) the mixed image video of the lesson (V); and (b) the video stimulated, post-lesson reconstructive student interview (I). These sources are identified in the text.

For the purpose of analysis, Lesson 12 was divided into episodes or intervals in time during which Dean focused on a particular idea. The first 15 of the 21 episodes in the lesson have been included in Table 1. Episodes 16-21 have not been included and will not be discussed because they relate to the development of the algebraic rule and Dean did not link this rule to his ideas developed earlier in the lesson. In Table 1, episodes are numbered consecutively in Column 1 according to the time at which the episode occurred (see Column 2). The context of the episode (Column 3), and a description of the episode (Column 4) are also included. Column 5 displays the cognitive elements of the process of abstraction observable for Dean (V & I). The 15 episodes included 5 off-task and 10 on-task episodes. Three of the off-task episodes (1, 10 & 14) were instigated by Dean who engaged in across-class whispered conversations with Cam in relation to her hat that had been confiscated from Dean. Of the 10 episodes where Dean focused on mathematics, he struggled to understand what was expected in Episodes 4, 7, 8, 9, and 11 (I & V). Dean demonstrated observable cognitive elements of the process of abstraction in Episodes 2, 3, 12, and 15, and there was insufficient evidence to detect Dean's cognition during Episode 6. As can be seen from Table 1, the critical intervals in time occurred in Episodes 12 and 15 where Dean began constructing new ideas. The earlier episodes are now briefly described in preparation for a more detailed analysis of Episodes 12 and 15. As a variety of activities occurred simultaneously during whole class talk and organized pair-work, indicators of involvement were used to determine the focus of Dean's attention. The types of activities that occurred simultaneously in this classroom included: (a) groups of students engaged in their own on-task or off-task talk during whole class-

talk; or (b) students engaged in talk in pairs or in larger groups whilst the talk of adjacent groups and the teacher (assisting other groups) was also audible.

<i>No</i>	<i>Time</i>	<i>Context of Episode</i>	<i>Episode description</i>	<i>RBC</i>
1	9:16-9:32	Across Class Off-Task Talk	The confiscated hat (i)	
2	11:16-13:52	Task Instruction	No. of sides of polygon	R
3	14:14-16:30	Task Instruction; Procedure	Make triangles by joining vertices	R
4	16:30-17.19	Investigation	No. of triangles in Polygons	
5	17:42-18:12	Small Group Off-Task Talk	Put Popeye Away	
6	18:53–20:40	Student responses to Col. 3.	No. Triangles in Polygons (ii)	
7	20:02-22:40	Task Instruction; Procedure	Sum of interior angles of polygon	
8	22:54-25:10	Simultaneous Work	Dean sorted work; Ted cut out triangles	
9	25:10-25:30	Simultaneous Work	Class discussed results; Dean struggled with procedure (Ep. 7)	
10	25:56-28:20	Across Class Off-Task Talk	The confiscated hat (ii)	
11	28:20-32:26	Small Group Instruction	Teacher demonstrated procedure (Episode 7) to Dean and Ted.	
12	32:26-33:48	Simultaneous Work	Dean attempted procedure; Ted found pattern; teacher assisted	R B C?
13	34:15-34:20	Small Group Off-Task Talk	The pot plant	
14	34:25-35:22	Across Class Off-Task Talk	The confiscated hat (iii)	
15	34:20-37:30	Whole Class Summary	Dean develops novel method	R B C?

**Key:** Cognitive elements: R, recognizing; B, building-with; C?, constructing interrupted

Table 1: The first 15 of the 21 Episodes in Lesson 12 from Dean's Perspective

In Episodes 2, 3, 4, and 6, the teacher explained the task by providing sheets: (a) a table with headings 'Name of Polygon', 'Number of Sides', 'Number of Triangles', and 'Sum of Angles'; and (b) named polygons with 3-10 sides. He demonstrated the procedure for sectioning each polygon into triangles—join all vertices of the polygon to a particular vertex. Students were required to fill in the columns 'Number of sides' and 'Number of triangles' which was Year 7 work for Dean (I). Dean was unsure how to fill out the third column of the table in Episode 4 as evidenced by his comment 'Oh I get it' when he heard an adjacent group explaining late in the episode (V). The teacher introduced the procedure for finding the sums of interior angles of polygons in Episode 7 by reminding students they knew the answer for the first polygon (a triangle). In an earlier lesson, the teacher had torn off the 'corners of the triangle' (teacher's wording) and licked them and placed them together on the board commenting: 'see they make a hundred and eighty'. In Episode 7, the teacher demonstrated a similar procedure with a quadrilateral—cut out the triangles, tear off their 'corners', place them together, and find the total angle. When asked why they were doing this (by a student), the teacher replied: 'to find the answer'. Dean's comments in class and his interview-reconstruction of Episodes 9, 11, and 12 demonstrated he did not know how to implement the teacher's procedure. Dean knew he

had to rip off corners but did not know what to do after that as illustrated by an excerpt from Dean's interview [Key: '{...}' dialogue omitted; '[' ]' researcher's comments]:

Dean I always put [pause] I didn't know [pause] where the corners went [pause] in it {...} I was doing it all different- I was facing them *out* ... and *up* {...}

Dean did not know to face the vertices of the angles towards the center and juxtapose them to see the total rotation. He had the angle vertices pointing many different ways. Dean's lack of understanding of the purpose of the teacher's procedure was further illustrated by Dean's response to a question from Simon who asked Dean why you tear not cut the corners. Dean shrugged initially, then when he overheard the teacher explain to an adjacent group: 'cut not tear so you know which are the corners', Dean turned to Simon and laughed: 'You have got to find which is the corner'. In Episode 11 Dean made an unsuccessful attempt to find the sum of the interior angles of a quadrilateral. The teacher then demonstrated the procedure for Dean: 'Right, tear, tear, tear, we get the pointy ends in together, and that gave us a hundred and eighty, straight line.'

Dean controlled the start of Episode 12 as he attempted the teacher's procedure with a pentagon. The teacher progressively took control by repositioning the 'corners' Dean had placed. The actions of the teacher assisted Dean to see that the teacher's procedure required the 'corners to face in' (V), but did provide Dean with reasons for why this was so (I). The teacher engaged in a parallel but separate dialogue with Dean's partner Ted who developed a numerical pattern: 'add a 180 each time'. Dean paid no overt attention to this parallel dialogue until the teacher asked Ted 'what's 360 plus 180?'. The teacher then used the 'corners' he had juxtaposed with Dean to evaluate Ted's response of 540° and exclaim: 'Aha it is'. Dean then looked from Ted's table, to the juxtaposed angles, to his own page. Dean did not attempt the teacher's procedure again after Episode 12.

Enriched transcripts and visual displays of Dean's abstracting process were generated for Episodes 12 and 15 but due to space constraints have only been included for Episode 15. Enriched transcripts include descriptions of Dean's body language and interview comments beside the relevant lines of transcript (Table 1). Episode 15 is now summarized and supported with the enriched transcript (Table 2) and a visual display (Figure 1) that differs in several aspects to the displays developed by Dreyfus, et al. (2001). The display used in this paper contains evidence of task involvement (see key to Figure 1), and every line of transcript is related to Dean's cognition rather than to the cognition of the speaker. In Figure 1, line numbers are listed down the left hand side with the inclusion of lower case letters after line numbers to indicate dialogue captured on the student (but not the teacher) microphone. The three groups of columns display Dean's cognitive activity (left), social elements of Episode 15 (center) and Dean's task involvement (right). In Lines 439 and 442, the 'I' and 'V' beside the task involvement columns show indicators drawn from interview and video data. Due to space limitations, task involvement indicators for interviews have only been shown for the lines where Dean's video-stimulated reconstructive interview data provided evidence not available from Dean's body language in the classroom video (Lines 439 and 442).

Line. Time	Dean's new method 34:53-37:30	Relevant Excerpts from Post-Lesson Video Stimulated reconstructive Interview
435. 35:47	T: // [wrote table headings on board] Those people who managed to work it out for the four-sided shape found it was 360. Around about this point Ted had an idea of what was going on. [Dean listened, wrote, looked at board].	<b>Dean:</b> {...} on the board it said [pause] two triangles [pause] which is [pause] 360° [pause] and then I just thought [pause] you keep each triangle you had [pause] 180° and then [pause] <i>so on</i> but ... yeah.
436/7 35:58	T: What did you think was going on Ted? Ted: One eighty you plus one eighty. [Dean watched board]	
438. 36:03 439. 36:04  439a)	T: Every time you add one eighty. So we did the five-sided shape and we got five hundred and forty. Which was one time around plus another half. [Dean watched the board]  Tessa: Oh so you always go up by one eighty.	<b>Dean:</b> {...} so a 180 plus 180 is 360- plus another hundred and eighty [pause] be 540. <b>Interviewer:</b> How did you know that? <b>Dean:</b> He said [pause] in a previous lesson [pause] each triangle adds up to 180- that was when we {...} the protractor [pause] and we did all those lines {...} once you get a triangle {...}. In a previous lesson [pause] hard to explain. I may have <i>known</i> that like in a <i>previous</i> lesson but [pause] I probably didn't think of it for [pause] because it's <i>another topic</i> [pause] like we were just doing <i>that</i> [pause] <i>then</i> {...} finding out the shapes and triangles {...} <b>Interviewer:</b> What does ... 'adds up to 180' mean? <b>Dean:</b> Um ... I'm not actually sure.
441. 36:17	T: // Every time you go up by one ... you keep adding up 180. [Dean looked at the developing table then wrote]	<b>Dean:</b> But there's another way you can do it as well {...} each triangle you get [pause] in the shape [pause] which is 180° [pause] so that would be 180, 180, 180 {...} 360, plus another hundred and eighty {...} 540.
442. 36:22	T: [to all] Sally had a six-sided shape // and she said I've got two three sixties. Which is 720. [Dean wrote in his book. When he finished his calculation he leant slightly towards Ted and Ted's page and pointed to Ted's page]	<b>Dean:</b> I think <i>that's</i> where I got it from actually [fast confident voice] {...} Hold on! [Rewinds video] {...} [slow pensive] <i>yeah hold on</i> [fast pace] [pause] see how that's got [pause] 1, and then 2 and then 3 and then 4 and then 5? [pause] Yeah wait on! [fast pace] {...} yeah I think that's where I got it from. See it goes up by 180 each time.
442a. 36:23	Dean: // That'll be ten eighty. [Dean looked at Ted]	

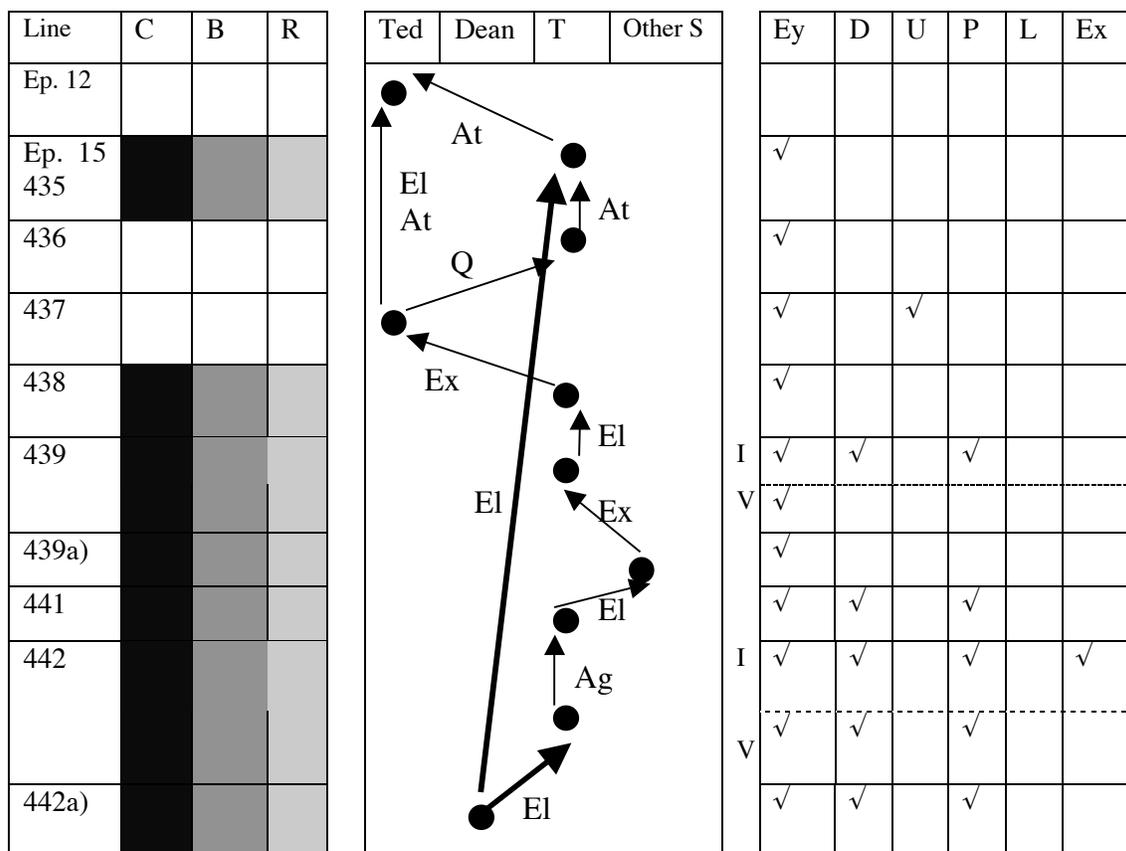
Key: '[' researcher comments; '/' simultaneous event; '{...}' dialogue omitted; 'italics' student emphasis. Numerical values rather than words have been used where a number was correctly worded.

Table 2. Enriched transcript for Episode 15: Dean's new method

**Dean's cognitive element involvement**

**Social elements**

**Dean's task**



**Key. Cognitive:** C, constructing; B, building-with; R, recognizing. **Social:** C, control; El, elaboration; Ex, explanation; Q, query; Ag, agreement; At, attention

**Involvement (video, interview):** Ey (eyes, respond to questions); D (point body or pen to task, try to answer fully); U (unaware, cut to add ideas); P (participate, few prompts needed); L (latch to each other's comments, cut in eagerness to answer); Ex (exclaim, pace and emphasis). 'Cut': interrupt interviewer's question. '—': aspects of the social interaction contributing to Dean's idea.

Figure 1. Visual display of Cognitive, Social, and Involvement for Episode 15

During Episode 15, Dean spoke once (Line 442) and attended selectively to classroom discussion. Where the class focused on Ted's pattern, Dean focused on the number of triangles and adding a new 180° for each triangle (Line 436, interview comments during Lines 435, 439, 441 & 442). Towards the end of the episode Dean softly told Ted his result from building-with his new method (Line 442a in Table 2 & Figure 1). During most of Episode 15, Dean paid attention to the table on the blackboard. He sometimes leaned over his work and wrote calculations (Lines 435-441). As the table on the board was progressively completed, Dean reflected about the number of triangles in polygons with 3-6 sides (I, Line 441). Dean built-with his newly developed sequence of procedures to find the sum of the interior angles in a polygon with more than 6 sides (Line 442a).

## DISCUSSION AND CONCLUSIONS

Dean self-instigated pursuit of a discovered complexity: 'the corners of each new triangle make another 180°' when the teacher placed the 'corners' of triangles in Episode 12. This led to self-directed exploration of links with the idea: 'angles in a triangle add to a hundred and 180°' (self-selected from another topic (Line 439, Table 2)). He horizontally reorganized a mathematical structure by using the number of triangles in each polygon (I: Line 435, Table 2) and successively applying the rule 'angles in triangles add to a hundred and eighty degrees' for each triangle (I: Line 441, Table 2). Dean's focus of attention on Column 3 on the board was self-directed; the class focused on numerical patterns in Column 4. Dean identified video around Line 442 (Table 2; 6-sided shape) as when his new idea crystalised (empirical generalization). Dean's perception of angles, that he was unable to identify in diagrams, as amorphous entities associated with polygons provided an inadequate cognitive artifact to scaffold the integration of ideas and gain insight from the relative positions of the interior angles of the triangles and the polygon (theoretical generalization). Nevertheless, Dean's ideas were more mathematical than those used by the class. Implications for practice include: (a) knowledge assumed to be simple and understood by all (like the idea of angle) may not be understood and could inhibit the development of a new more complex concept; and (b) students could benefit from tasks that provide opportunities to develop assumed knowledge and more complex concepts simultaneously. The interesting theoretical question for further study is: What are consequences of the subsequent development of an inadequate cognitive artifact (from an empirical generalization to a theoretical generalization) in relation to any related more complex empirical generalization the student may have already developed?

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