INFLUENTIAL ASPECTS OF DYNAMIC GEOMETRY ACTIVITIES IN THE CONSTRUCTION OF PROOFS

Ernesto Sánchez & Ana Isabel Sacristán
Center for Research and Advanced Studies (Cinvestav), Mexico

We present results from a study which investigated the influence of dynamic geometry-based activities in the development of proving skills (in geometry) in high-school students (15-16 years of age). After a 12-week course on Cabri and the writing of conjectures and proofs, students were asked to write and prove conjectures based on their observations in six Cabri-based activities. We analyzed the written data in the light of Balacheff's work (1987, 1999). Although progress is shown on the level of the development of knowledge, in that the objects and their relationships become more meaningful, there are still many difficulties in decontextualizing the activities and in the development of a functional language, necessary for the passage from pragmatic to intellectual tools.

INTRODUCTION AND THEORETICAL FRAMEWORK

For several years we have been investigating the role that new technologies play in the stages of the development of a proof in mathematics education. The use of technological tools bring the possibility for different types of conceptualizations of mathematical objects which may help or hinder the processes involved in the development of proofs. Most of the research regarding the use and influence of technological tools in proving has been done with regards to dynamic geometry environments (Hoyles & Jones, 1998; Balacheff, 1999; de Villiers, 1998, 2002; Jones, 2000), and in this paper we will present data from a study involving dynamic geometry-based activities for the construction of geometrical proofs; however, it is worth noting that it not all the research is exclusive to geometry or to that tool (e.g. Sacristán & Sánchez, 2002).

Some researchers have called for using proofs to create a meaningful experience; that is, as a means to help students understand why results are true (Hanna,1998), by providing an opportunity for exploration, discovery, conjecturing, refuting, reformulating, explaining, etc. which technological tools such as dynamic geometry environments can provide (De Villiers, 2002). Computer tools can be used to gain conviction through visualization or empirical verification, but as De Villiers (2002) points out, proofs have multiple functions that go beyond mere verification and that can also be developed in computer environments: such as *explanation* (providing insight into why it is true), *discovery* (the discovery or invention of new results), *communication* (the negotiation of meaning*), intellectual challenge* (the self-realization/fulfillment derived from constructing a proof), *systematization* (the organization of various results into a deductive system of axioms, concepts and theorems).

Balacheff (1987, 1999), in particular, has introduced certain ideas that we consider important for our analysis. First of all, he distinguishes between the terms *explanation,* and two terms which are generally both translated as *proof* into English: the French terms *preuve* (proof) and *démonstration* (mathematical proof) to . For him, an explanation is a discourse that aims to clarify the true makeup of a proposition or result. *Preuve* (proof) is an explanation that can be accepted by the community and which leads to the existence of a common system of validation. Finally the *démonstration,* or mathematical proof, is an accepted proof by the community of mathematicians (Balacheff, 1987). Balacheff has also proposed a distinction between *pragmatic proofs* and *intellectual (*or *conceptual) proofs*; emphasizing the role of language in the passage from the former to the latter. Pragmatic proofs are those based on effective action carried out on the representations of mathematical objects. They lead to practical knowledge that the subject can use to establish the validity of a proposition. Intellectual proofs demand that such knowledge is reflected upon, and their production necessarily requires the use of language that expresses (detached from the actions) the objects, their properties and their relationships. In other words, pragmatic proofs are based on action, while the use of a functional language (which includes a specific vocabulary and symbolism) and a "mental experience" (where actions are interiorized) characterize the transition to the intellectual ones. The transition from pragmatic proofs to intellectual proofs culminating in mathematical proof, involves three components: the *knowledge* or *levels-of-action component* (the nature of knowledge: knowledge in terms of practices — "*savoir-faire";* knowledge as object; and theoretical knowledge); the *language* or *formulation component* (ostentation, familiar language, functional language, formal language); and the *validation component* (the types of rationale underlying the produced proofs: from pragmatic, to intellectual, to mathematical proofs).

The development of a functional language involves processes of "detemporalization" and "decontextualization". That is, it should be a language for talking about mathematical objects and for communicating ideas related to them, independently of the situation, school context, or of the persons with whom the communication takes place (e.g. the teacher).

In the next section, we present results from a teaching experiment using Cabri-Géomètre, that is part of our research in investigating the role that dynamic geometry environments can have in the passage from pragmatic to intellectual proofs in geometry.

A TEACHING EXPERIMENT USING CABRI-GÉOMETRE

The aim of the research presented here, was to investigate alternative types of activities, in this case using dynamic geometry software, for improving proving skills in geometry in high-school students. (It is worth noting, however, that in two other parts of our research we have also investigated the same activities with teachers, and a group of students different than the one reported here). To select our subjects, we applied a diagnostic questionnaire, based on the questionnaire devised by Healy & Hoyles (1998), to a group of 40 Mexican high-school students of ages 15-16 years, in order to evaluate their understandings of aspects linked with mathematical proofs. We marked the correct answers in order to get a general evaluation of each student according to the number of right answers. The 8 top students from the results (which in this case coincided with the students that their teacher considered the best in mathematics) then participated in an experimental 35-hour course consisting of 3 hour-sessions once a week. These students, as in general students of this educational level in Mexico, have studied some themes of Euclidean geometry, but not a full course.

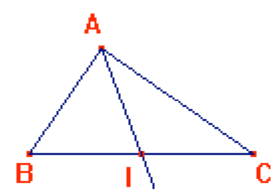
The aims of the course were to instruct students in three aspects: (i) the use of the Cabri-Géomètre software (knowledge of commands, construction of objects, etc.); (ii) a review of basic geometrical knowledge (simple propositions on parallel lines, angles and triangles); and particularly (iii) the writing of *conjectures* (with an emphasis of having students recognize that a conjecture is a proposition formed by two parts: an antecedent and a consequent) aiming to have students learn to express explanations, verifications and, if possible, proofs of geometrical theorems and properties. We are aware of concerns that explorations in dynamic geometry environments, where conjectures are confirmed by testing the figures through dragging may reduce the perceived need in students for deductive proof (Chazan, 1993; Hanna, 1998; Hoyles & Jones, 1998), and that is one of several reasons why we placed emphasis on the third aspect of the instruction process.

The didactical design of the course consisted in activities similar to those reported in this paper: the teacher explained how to construct objects in Cabri, let the students explore them freely, then there was a joint discussion on the writing and proving of conjectures derived from the activity. At the end of each session, students were given an activity to carry out on their own. During the last session, students were asked to work on six activities, and for each one write and prove a conjecture.

Each of the activities (all related to medians and areas of triangles), was meant to lead into a specific proposition. These propositions constitute a system in the sense that they can be deduced from previous propositions. The background needed to construct the proofs was simply the formula for the area of a triangle (base x height /2) and propositions constructed or derived from the system of six activities. In this paper we will only talk of the first two activities as they are representative of all the others.

Activity 1

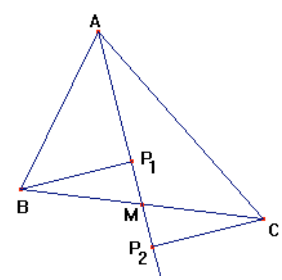
• Draw a triangle, label the vertexes with A, B, C. • Mark the midpoint of segment BC; call it I. • Define the triangles △AIB and △ACI. • Get the areas of the triangles △AIB and △ACI. • Grab any vertex A, B, or C and drag it to new position. Observe what occurs with the areas. • Write a *conjecture* about your observations and write the corresponding proof.



Activity 1 was designed to lead to the construction of a proposition related to the theorem: *If in the triangle △ABC, AM is a median, where M is the midpoint of BC, then Area (△AMB)=Area (△ACM).*

Activity 2

• Draw a triangle △ABC (label the vertexes). • Mark the midpoint of segment BC (call it M). • Draw the ray AM • Draw the perpendicular line to AM that passes through the vertex B • Define the point of intersection between AM and the above perpendicular; call it P_1. • Draw the perpendicular line to AM that passes through the vertex C • Define the point of intersection between AM and the above perpendicular; call it P_2. • Hide the perpendicular lines BP_1 and CP_2 • Draw the segments BP_1 y CP_2 • Get the length of the segments BP_1 y CP_2 • Drag the vertexes A, B, or C and observe what happens with the lengths of the above segments • Write and prove the *conjecture* about your observations.



Activity 2 was designed to lead to the construction of a proposition related to the theorem: *If in a triangle △ABC, AM is a median, then the distance from B to AM is the same as the distance from C to AM.*

SAMPLE DATA AND INTERPRETATION OF SOME RESULTS

We used the written propositions of the students as a window to get insights into their abilities to understand the fundamental ideas involved in the proof, as well as evaluating their skills to formulate and write conjectures and proofs. From a detailed analysis of the propositions written by the students we deduce that, on the one hand, through the exploration activities in the Cabri environment, students do develop a certain "phenomenological" knowledge of the elements and notions involved; this shows some progress on the level of the knowledge component. On the other hand, the written data also shows the difficulties that students have in expressing the propositions; this suggests little progress on the level of the language component. Most of the "proofs" that students gave were really *explanations*, since few of them could be classified on the level of intellectual proofs; all the others would be classified as pragmatic proofs.

The first example is from a student called Alexandra:

> **Alexandra's Proposition - Activity 1**: In every triangle, when drawing the [median[1]] the height and the bases will measure the same and therefore will have the same area. Since we found the midpoint for BC, the two bases will measure the same and because the height is perpendicular to the base, both triangles will have the same height.

Anyone unfamiliar with the activity would be unable to understand it. There is an absence of a reference to the two triangles that get formed when sketching the median of a triangle; this makes the statements "the height and the bases will measure the same and therefore will have the same area" very confusing. She also uses the symbols (she uses the reference BC for the base of the triangle) referred to in the activity, but doesn't properly introduce them in her proposition. There are similar problems in her proposition for the second activity:

> **Alexandra's Proposition - Activity 2**: In a triangle where we draw the [median[2]], the perpendicular segments to it will measure the same if they are the height of the triangles that are being formed. In the triangle ABM the segment BP_1 is its height because it is perpendicular to the base in the same way for the triangle AMC.

This second proposition is even more confusing than the first: it mentions the segments perpendicular to the median, but it doesn't specify that they each pass through one of the two remaining vertexes. Nevertheless, in spite of the problems with the written proposition, it is worth noting that Alexandra was one of only two students who was able to link the result of the first activity with this second one: in her "proof" she seems to be using the congruence of the area of the two triangles that are formed with the median. In a previous study to the one reported here, none of the students linked the two activities in this way. Independently of that, it is clear from her written work that she has a good understanding of the content of the proposition she makes, and although her statement is

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[1] The quote has been translated from the original Spanish. In Spanish, Alexandra used the term *mediatriz* which is the perpendicular bisector, but she was referring to the median because that is what she constructed in the activity. Nevertheless, her mistake shows a confusion (very common due to the similarity) between the terms *mediana* and *mediatriz* (perpendicular bisector).

[2] As in the previous activity, Alexandra used the term *mediatriz* to refer to the median.

faulty, she does present the correct elements for the proof. Like Alexandra, most students were able to identify the elements and relationships needed for the proof, but they were unable to properly articulate them in their written propositions; nevertheless this is important in making the activities meaningful (progress on the level of the knowledge component).

In Alexandra's case there is also an attempt to use symbolic references and to generalize her propositions ("in every triangle"). However, although attempts to generalize were found in most students, the use of symbolic notations wasn't as common or was misused (see further below), and students very often relied on familiar language.

In fact, students in general didn't pay much attention to be rigorous in the writing of the conjectures they observed. They centered more on the explanation and "proving" than on clarifying the propositions they discovered through the Cabri-based activities. This is in spite of the fact that during the instruction phase the emphasis was put on the writing of the conjectures. In the written work from Activity 1: only three of the eight students made any attempts of writing down the proposition, and only one of those three cases can be considered almost correct (there was no reference to the original triangle). All the other students wrote directly an explanation or a proof. In the second activity the situation is even worse: only one student (Alexandra) tried to give a complete proposition with the problems already discussed; three other students only gave the conclusion without stating the conditions and hypothesis of the proposition; all of the other students directly wrote an explanation or proof. It is also worth noting that most of the students (see sample data) write both their proposition and their proof in a single paragraph, again indicating that their focus was more on explaining than on writing a formal proof.

> **Zaira's Proposition - Activity 2**: The length of P_1 is equal to P_2, because both are perpendiculars to AM with respect to a point, but M is the midpoint of the triangle and because of that it is the same measure that goes from M to B than the one from M to C.

Before the first coma, she tries to state the "result" (that the segments are equal), followed by a failed attempt to prove it. We suggest two reasons why students avoid writing their conjectures:

- The difficulties in describing and articulating the geometric elements that come into play
- The school setting: i.e. the habit of having the activity given by the teacher and knowing that "he/she knows what is being talked about." It is therefore enough to just give some indication that the activity was carried out and understood.

An aspect which points to the difficulty in the appropriation of a functional language is the fact that Zaira uses the symbols P_1 y P_2 in reference to the segments BP_1 and CP_2 (Barbara does the same); this suggests that Zaira sees the symbols only as labels that can simplify the writing ignoring how they can refer to the structure of an object and its properties. (In this sense, another student used MP not in reference to a segment MP in the figure, but simply as an abbreviation for the term midpoint).

However, it is important to mention that there may also be a cultural dimension to the difficulties that students have in the writing and formulating of the propositions. General writing difficulties have been found almost everywhere in the world, and we shouldn't be surprised that even more difficulties arise in the writing of conjectures and proofs. It is evident from the written work of the students as described above, that these students are not used to expressing mathematical ideas (whether formally or informally) despite the emphasis given to that in the teaching experiment. But it is important to note that in the Mexican Mathematics School Curriculum there are no explicit objectives to teach proof. Furthermore, although we have yet to substantiate the following statement, we have observed a general lack of encouragement in schools for students to express mathematical ideas with rigor and formality: the emphasis tends to be simply on producing the correct solution. Thus the above deficiencies are even less surprising, and cultural aspects may enhance this problem.

Problems of decontextualization

In the theoretical framework we mentioned the importance of decontextualization for the development of a functional language. We found problems of decontextualization in students in two ways: first, as mentioned above, in that they assume that the reader of their propositions (the teacher) knows what they are talking about. There is another level of "situatedness" which is more specific to the dynamic geometry environment. Take, for example, Barbara's proposition:

> **Barbara's Proposition - Activity 1**: The triangle has the same base and the same height and because it is divided in two by the median, thus when I move one of the points of the triangle they always have the same measure for their area.

It is worth noting that as the problems become more complex, references to actions carried out with the software (e.g. "if I drag…", "when I move…") become more and more common. This could be interpreted that as the problem is more complex, the more difficulties students have in the decontextualization. We dare say that these "situated" propositions by the students are akin to what Noss & Hoyles (1996) have called "situated abstractions" where the generalizations that occur are still in terms of the actions and the situation in which they arise. In terms of Balacheff's theories, we would say that they are still on the level of pragmatic proofs.

The construction of meaning through the DGE activities

Nevertheless, we find that the use of the dynamic geometry software, helped students make progress on the level of the knowledge component. Students seem to grasp the meaning of some of the theorems thanks to the phenomenological approach that the use of Cabri makes possible. There is a fundamental difference in the construction of the geometrical figure between doing it with paper-and-pencil and doing it in a dynamic geometry environment: whereas in the first one it is the construction of a particular case, in the latter one it is actually the construction of a "general case". The construction in Cabri also requires the definition of structural elements (e.g. the midpoint is intrinsically defined as the midpoint); failure to do so will result in the figure "falling apart" through dragging. In the first activity, for example, by dragging and experimenting they can observe on the screen that the areas of the two triangles that get formed when dividing a triangle with a median, are always equal: this gives students a clear reference of what the proposition is about. When they are asked for a proof, they look for an explanation and they are generally able to observe the necessary elements that are needed for the proof, suggesting the fundamental idea of the proof.

In terms of the theoretical framework and Balacheff's ideas, we can say that the Cabri-based activities seem to favor the knowledge component, while the development on the level of language and formulation remains a difficult area. The benefit of the software seems to be in that the construction activities produce appropriate meanings for the objects and their relationships, but progress on the level of the validation component depends on the other two components: both the knowledge and the language ones.

The written work that we have analyzed in particular in this paper, reveal the ways in which some of the contextual elements involved in the implementation of the tasks are difficult to abstract in order to produce an approximation to, not even a written proof, but also to a modest written mathematical explanation of the geometrical propositions. The students are still far from being able to produce written texts explaining the results observed in an activity but that are independent of the situation of the activity.

The role of language is crucial in the intellectual work involved in geometry and our research points to its complex nature. Geometrical activities involve two types of treatments of the geometrical objects: the visual treatment and the discursive treatment. The software favours the visual treatment, since it helps avoid the discursive treatment and use of symbolic language; this in turn, by permitting an immediate access to "concrete forms of abstract entities" (Laborde & Laborde, 1995, p. 243), helps develop a phenomenological knowledge of the mathematical objects in a much faster way that in paper-and-pencil activities.

But, on the other hand, the theoretical knowledge of geometry, that which supports a mathematical proof, is developed in the transition between pragmatic and intellectual proofs, and demands the development of a functional language. It seems that the software-based explorations, without complementary activities specific to the development of language, do not provide many opportunities for the development of first a descriptive language and then a functional one.

We consider this one of the limitations of dynamic geometry environments where objects are controlled through the mouse, as opposed to other computer tools, such as Logo, where the objects are controlled through the symbolic notation of the programming code.

Finally, although a profound discussion of this issue is beyond the possibilities of this paper, it is important to mention a further problem that we have identified in the use of Cabri. We find that an important activity in the interpretation of a figure in order to discover a conjecture and deduce its proof, is that which consists in the distinction and regrouping of elements or subfigures of a given geometrical figure; what Duval (1995) calls *reconfiguration*. Because of the sequential way in which figures are constructed in Cabri, the process of reconfiguration is not facilitated, as we have observed in all our research subjects. We believe this may be a reason why most students did not link the propositions derived from one activity with the others, as Alexandra did (see above).

Despite the limitating influences of the use of Cabri in the development of functional language and in facilitating reconfiguration, we believe that the advantages in creating meaningful knowledge provide an important foundation for the passage and understanding of mathematical proofs and deductive reasoning. And further research may help develop complementary activities to assist students overcome those limitations.

References

Balacheff, N. (1987) Processus de preuve et situations de validation. *Educational Studies in Mathematics* 18, 147-176

Balacheff N. (1999) Apprendre la preuve. In: Sallantin J. & Szczeniarz J.-J. (eds.) *Le concept de preuve à la lumière de l'intelligence artificielle* P.197-236. Paris: PUF.

Chazan, D. (1993) High School Geometry Students' Justification for their Views of Empirical Evidence and Mathematical Proof, *Educational Studies in Mathematics* 24 (4). 359–387.

de Villiers, M. (1998) An alternative approach to proof in dynamic geometry, in R. Lehrer and D. Chazan (eds) *Designing Learning Environments for Developing Understanding of Geometry and Space*, Lawrence Erlbaum Associates, Hillsdale, N.J., pp. 369–393.

de Villiers, M. (2002) Developing Understanding for Different Roles of Proof in Dynamic Geometry. Paper presented at ProfMat 2002, Visue, Portugal, 2-4 October 2002.

Duval, R. (1995) *Sémiosis et pensée humaine. Registres sémiotiques et apprentissages intellectuels*. Peter Lang, France.

Hanna, G. (1998) Proof as understanding in geometry. *Focus on Learning Problems in Mathematics* 20(2&3), p.4–13.

Healy, L. & Hoyles, C. (1998) *Justifying and proving in school mathematics. Technical report on the nationwide survey*. Mathematical Sciences. London: Institute of Education, University of London.

Hoyles, C., Jones, K. (1998) Proof in dynamic geometry contexts, in C. Mammana, V. Villani, (eds) *Perspectives on the teaching of geometry for the 21st century,* Kluwer Academic Publishers, The Netherlands.

Jones, K. (2000) Providing a foundation for deductive reasoning: Students' interpretations when using dynamic geometry software and their evolving mathematical explanations. *Educational Studies in Mathematics* 44: p.55–85. Kluwer Academic Publishers.

Laborde, C.; Laborde, J. M. (1995) What about a learning environment where Euclidian concepts are manipulated with a mouse? In A. A. diSessa, C. Hoyles, R. Noss & L. D. Edwards, *Computers and Exploratory Learning,* Springer-Verlag, Germany.

Noss, R. & Hoyles, C. (1996) *Windows on Mathematical Meanings. Learning cultures and computers*. Kluwer Academic Press.

Sacristán, A. & Sánchez, E. (2002) Processes of proof with the use of technology: discovery, generalization and validation in a computer microworld. In Cockburn, A. D., Nardi, E. (eds) *Proceedings of the 26th Conference of the International Group for the Psychology of Mathematics Education*, Norwich, UK, July 2002. Vol. 4, pp. 169-176.