TO PRODUCE CONJECTURES AND TO PROVE THEM WITHIN A DYNAMIC GEOMETRY ENVIRONMENT: A CASE STUDY

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This paper analyses a case study of a pair of students working together, who were asked to produce conjectures and to validate them within the dynamic geometry environment Cabri. Our aim is to scrutinize the students' reasoning, how the gap from perception to theory is filled, how Cabri influences the reasoning. We have singled out a sequence of phases in which the students pass from exploration to increasing degrees of formal reasoning. Our study reveals, among other things, that Cabri fosters the flexible use of methods close to analysis and synthesis.

INTRODUCTION

In this paper we report a case study concerning a pair of students who explored an open problem within the dynamic geometric environment Cabri-Géomètre I (henceforth Cabri). This activity has brought to the fore a variety of strategies and ways of thinking that makes it possible to detect and to analyze how the students came to prove. By scrutinizing the students' reasoning we single out continuity and leaps in the transition from perception (observation of figures in the screen) to theory.

To explain the context in which the studied activity has developed we state our ideas on proof in classroom. In this concern a recent paper by Herbst (2002) stresses two main issues:

> As proof is intimately connected to the construction of mathematical ideas, proving should be as natural an activity for students as defining, modeling, representing, or problem solving. Yet, important questions that must be raised concern what it takes to organize classrooms where students can be expected to produce arguments and proofs and what proof may look like in school classrooms. (p.284)

In analyzing the *two-column proving custom* Herbst (2002) complains that, under this method, "proving activities for students have often been closer to exercising logic to validate obvious and inconsequential statements […], than to building compelling arguments for the reasonableness of important mathematical ideas […]." (p.284). This criticism may be applied also to other methods of teaching proof practiced in classroom, which are dominated by a rote learning model. These methods push students towards schema of proving such as those classified by Harel and Sowder (1998) as "ritual", "symbolic", "authoritarian". These schemas are far from being suitable to make the activity of proving a meaningful activity.

Our paper refers to an approach aimed at promoting a smooth transition from argumentation to proof, see (Furinghetti et al., 2001). The elements characterizing this approach are the following:

- focus on the debate about the construction of theorems and proofs, with the distinction between the problems of conforming to the standards of exposition and rigor and those of construction, validation, and acceptation of a statement
- specification of the rules for stating a theorem and proving it

- reflection on the environments which seem to foster the production of hypotheses and their formulation according to logical connection
- possibility of singling out cognitive continuity between the processes of producing and exploring the statement of a theorem and the construction of its proof, with particular attention to the reference theories and the leaps inside a theory and among different theories
- the role of the social dimension of the learning as for knowledge on theorems and proof, with particular reference to mathematical discussion in classroom and the modes of using various mediators (history, technology,…).

We think that only through experimenting personally the construction of parts of a theory (under the guidance of the teacher and in situations carefully projected) students may give up, when necessary, the perceptive level and appreciate the meaning of theories. To make students to construct parts of a theory means to allow them to experience the construction of mathematical knowledge at different levels: the level of exploring within particular cases, those of observing regularities, of producing conjectures, of validating them inside theories (which may be already constructed or in progress). In developing this approach we are concerned with the transition from elementary to advanced mathematical thinking. Gray et al. (1999) have pointed out that the "didactical reversal – constructing a mental object from 'known' properties, instead of constructing properties from 'known' objects causes new kinds of cognitive difficulty." (p.117)

Nunokawa (1996) has discussed the application of Lakatos' ideas to mathematical problem solving. In our approach to proof we are thinking something similar. We see students as immersed in a situation similar to that termed by Lakatos (1976) *pre-Euclidean*, that is to say a situation in which the theoretical frame is not well defined so that one has to look for the 'convenient' axioms that allow constructing the theory. The didactical suggestion implicit in Lakatos' words is that it is advisable to recover the spirit of Greek geometers. When they made proofs they were not inside a theory in which axioms were explicitly declared. Initially antique geometry developed in an empirical way, through a naïve phase of trials and errors: it started from a body of conjectures, after there were mental experiments of control and proving experiments (mainly analysis) without any sure axiomatic system. According to Szabo, this is the original concept of proof held by Greeks, called *deiknimi*. The *deiknimi* may be developed in two ways, which correspond to analysis and synthesis. These ideas suggest a way of realizing cognitive continuity in our approach to proof in classroom. Also they suggest the means to reach this objective: socialization, discussion, sharing of ideas.

REALIZATION OF OUR APPROACH TO PROOF

The general ideas we have previously discussed need to be adapted to the classroom needs as well as to the present conditions of students' learning. This requires creating environments suitable to exploration, production of conjectures, validation of these conjectures. To this purpose we propose to students open problems. We take as characterization of open problems the following, see (Arsac et al., 1988):

- The statement of the problem is short, so that it can be easily understood, it fosters discovery and all students are able to start the solution process.
- The statement of the problem does not suggest the method of solution, or the solution itself, but it creates a situation stimulating the production of conjectures.

- The problem is set in a conceptual domain which students are familiar with. Thus students are able to master the situation rather quickly and to get involved in attempts of conjecturing, planning solution paths and finding counter-examples in a reasonable time.

We think that open problems promote the devolution of responsibility from the teacher to students. This is even truer when students work in group and participate to classroom discussion. This situation fosters creativity, e.g. the ability to overcome fixations in mathematical problem solving and to produce divergent thinking within the mathematical situation (fluency and flexibility), see Haylock (1987).
Another element characterizing our approach to proof is the use of Cabri. It is widely recognized that the exploration with this kind of software amplifies the potential of producing conjectures, see (Santos-Trigo & Espinosa-Perez, 2002). At the same time it stimulates to prove the validity of the produced conjectures, see (Arcavi & Hadas, 2000).

In the situation we have outlined the statements to be proved are not provided by an authority (teachers, books), but are the result of an autonomous research and it is Cabri which confirms that a conjecture produced by students is 'good'. Thus the motivations to prove are different from those found in the usual didactical situations, where the task given to students is on the form "prove that…» The motivations we provide are similar to those of mathematicians at work, see (Burton, 1999).

THE CASE STUDY OF ALEX AND LUCA

The experiment reported is an example of what may happen in classroom when our approach is proposed. The class begins with a work of exploration and observation, which leads to produce conjectures. The validation of these conjectures is performed through the dragging text with Cabri. The way of reasoning is similar to that employed in empirical sciences, e.g. induction, abduction, analogy. In this context the role of proof is to *explain* why the produced conjectures hold within a theory (in our case Euclidean geometry).

The experiment was carried out in a class of a Scientific Lyceum (an Italian high school with a scientific orientation), at the beginning of the school year. The students (17 years old) worked in small groups (2 or 3 persons per group, 8 groups) with one computer per group. They were acquainted with exploration of open problems and had worked in group quite regularly before the experiment. They mastered Cabri. The time allowed for the experiment has been one hour and a half. In the following we describe the main phases of the work of the pair composed by Alex and Luca. The report is based on fieldnotes taken by the teacher (who acted as an observer) and on the students' protocols.

The statement of the problem was given without the figure. Alex and Luca draw quickly and accurately the quadrilateral *ABCD* and afterwards the quadrilateral *HKLM* using Cabri, see Fig. 1 (all figures made by the students with the computer were in color).

The problem:

You are given a quadrilateral *ABCD*. Consider the bisectors of the four interior angles: be *H* the intersection point of the bisectors in $\angle A$ and in $\angle B$, *K* the intersection point of the bisectors in $\angle B$ and in $\angle C$, *L* the intersection point of the bisectors in $\angle C$ and in $\angle D$, *M* the intersection point of the bisectors in $\angle D$ and in $\angle A$.
Investigate how *KHLM* changes in relation to *ABCD*? Prove your conjectures.

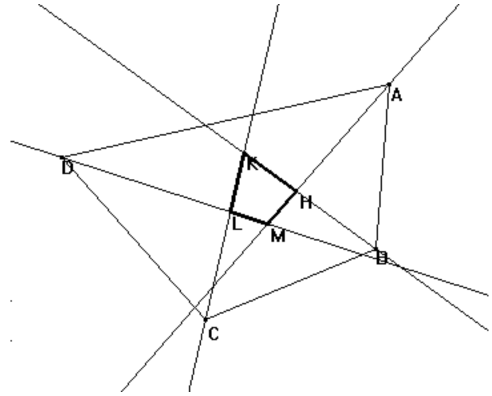

Fig. 1

PHASE 1

The students drag the vertexes *A*, *B*, *C*, *D* at random. This is the mode of dragging called "wandering dragging" by Arzarello et al. (2002): it is used when one is looking for ideas. This mode may be seen as almost static, since students drag the figures for a while and afterwards focus on the obtained figures kept still. During the wandering dragging they find the configuration reported in Fig. 2 in which the points *H*, *K*, *L*, *M* are almost coincident.

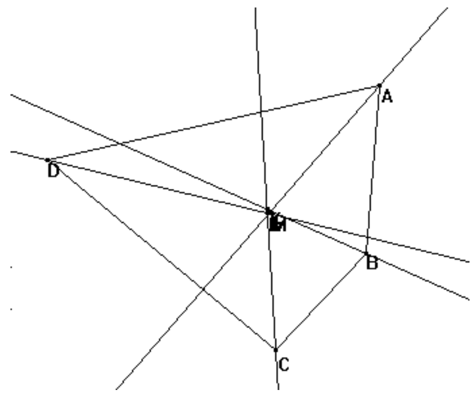

Fig. 2

PHASE 2

The mode of dragging changes significantly. Alex and Luca decide to focus on the internal quadrilateral *HKLM*. Of course, they can only act on the vertexes *A*, *B*, *C*, *D*, but they choose a particular configuration of *KHLM* (a point, a square, a rectangle, a rhombus, a parallelogram, a trapezium) and afterwards drag the vertexes *A*, *B*, *C*, *D* so that the quadrilateral *KHLM* keeps the particular configuration they have chosen. The students report only this part of the exploration in their protocol:

> To observe the changes of the figure we have considered the internal quadrilateral; that is to say first we have observed the particular cases of the internal quadrilateral and for each case we have looked at the changes of the external quadrilateral. With this method we have realized that different external figures correspond to each particular case of the internal figure: for example, when *H*, *K*, *L*, *M* are coincident, the external figure may be a square, a right-angled trapezium, or other figures.

These students are the only ones in the classroom using this mode of dragging based on the internal quadrilateral. This allows them to see very soon that not only squares and rhombuses generate internal quadrilaterals, which are points (In the case of squares and rhombuses the bisectors of opposite angles are coincident). The function of Cabri in this phase is to support transformational reasoning. We recall that Simon (1996) describes transformational reasoning as

> the mental or physical enactment of an operation or set of operations on an object or set of objects that allows one to envision the transformations that these objects undergo and the set of results of these operations. Central to transformational reasoning is the ability to consider, not a static state, but a dynamic process by which a new state or a continuum of states are generated. (p.201)

We stress that this phase marks a *leap* in the exploration. The mode of working goes back from the final result (a particular configuration of the internal quadrilateral) to the premises (the given quadrilateral and the bisectors of angles). This recalls the method of analysis, which is considered by many authors an efficient method of discovery. This method dates back to Plato and Greek mathematics. Hintikka and Remes (1974) describes it as follows:

> For in analysis we suppose that which is sought to be already done, and we inquire from what it results, and again what is the antecedent of the latter, until we on our backward way light upon something already known and being first in order. And we call such a method analysis, as being a solution backwards. In synthesis, on the other hand, we suppose that which was reached last in analysis to be already done, and arranging in their natural order as consequence the former antecedents and linking them one with another, we in the end arrive at the construction of the things sought. And this we call synthesis". (p.8).

Smith (1911) explains analysis as a method to solve problems and to prove theorems. He says that this method has several forms, but the essential feature

> consists in reasoning as follows: "I can prove this proposition if I can prove this thing; I can prove this thing if I can prove that […] until comes to the point where is able to add, "but I *can* prove that." This does not prove the proposition, but it enables [the student] to reverse the process, beginning with the thing he can prove and going back, step by step to the thing that he is to prove. Analysis is, therefore, the method of discovery of that way in which he may arrange his synthetic proof. (p.161-162)

Analysis leads to a construction and synthesis shows the validity of this construction. We note that schemas based on the triad analysis-construction-synthesis are present in the works of ancient mathematicians; in particular, on the method of analysis is based the development of modern algebra carried out in Viète's *In Artem Analyticem Isagoge* (1591). Gusev and Safuanov (2001) have argued that to solve problems requires various aspects of analytic-synthetic activities, and, in particular, analysis through synthesis.

PHASE 3

This is a phase of reflection made by students on what has been observed with Cabri. The students stop to explore. Through the reconstruction of some interesting configurations that they have obtained with Cabri they look for invariants. They write:

> We have looked for a relation among the figures obtained by means of the same internal figure. In this search we have discovered a theorem that we have formalized in this way: When all angle bisectors intersect in a point *P*, this point is the center of the circle inscribed in the quadrilateral.

In experiments with other students we have found that after having produced a conjecture the students continue to use Cabri to make figures which may prove the correctness of their conjecture. Our students show a different behavior: they are already convinced about the correctness of their conjecture and use Cabri only to refresh what they have done. The computer screen is no more an environment in which to conduct exploration and to take inspiration for conjecturing; it becomes a kind of fieldnotes keeper.

Our students show to be aware of the reverse path followed in their reasoning ("We have looked for a relation among the figures *obtained by means of the same internal figure* [italic added]"). Since they are convinced of the validity of their conjecture they are motivated to answer the question "Why the conjecture is valid?" Alex and Luca use the words "theorem" and "formalized", which evidence that they have definitely put themselves inside the theoretical framework of Euclidean geometry. They seem to perceive the function of proof as a process suitable to explain *why* a given conjecture is true. There is one sentence in their writing that shows the interlacement between the exploration ("In this search [made with Cabri]") and the theory ("we have discovered a theorem").

PHASE 4

Alex and Luca are ready to prove the conjecture produced. They abandon Cabri and use paper and pencil, also for drawing the figure on which their proof is based. [Fig. 3 reproduces accurately the drawing made by students with paper and pencil].

The proof is not complete and precise, but it may become acceptable with few amendments. In this phase it is clear that the mode of communication is changed. The focus has shifted from the facts observed in the screen to their justification in Euclidean geometry. The transition from the computer to paper and pencil marks the transition to the synthetic mode of proving (via the construction of Fig. 3).

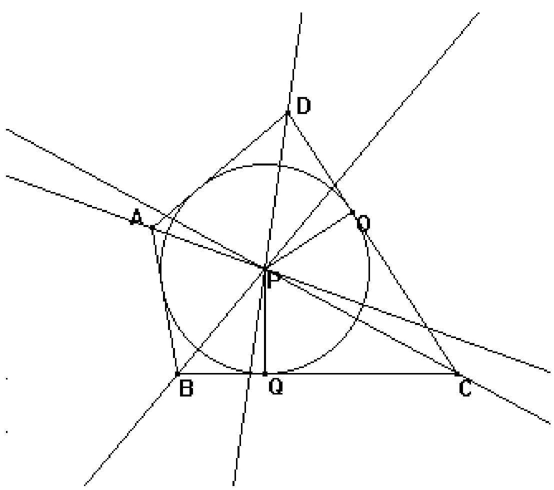

Fig. 3. (Made by students with paper and pencil)

We try to prove: to this aim we use two straight lines through the center which are perpendicular to two adjacent sides of the quadrilateral.

Hypothesis: PQ □ BC PO □ DC □ OCP = □ QCP

Thesis: $PQ = PO$ (radius of the circle)

Proof

We consider the triangles POC and PQC. We must prove that they are congruent. We know that the angles □ PQC and □ POC are right and congruent; also we know that □ QCP and □ OCP are congruent. Since PC is common to the two triangles, POC and PQC are congruent for the fourth criterion of triangles congruence. In particular, PQ and PO are congruent, so that they are two radiuses of the circle inscribed in the quadrilateral *ABCD*.

PHASE 5

Alex and Luca have sketched their proof. Since the statement of the problem given to them required studying the variation of *HKLM* in relation to the variation of *ABCD*, they change the statement just proved by them to emphasize the relation of dependence

We take a step backward: we have observed that the only common element among the figures obtained through a particular configuration of *HKLM* (*H*, *K*, *L*, *M* coincident) is the theorem that we have just proved. We know the theorem stating that a quadrilateral may circumscribe a circle when the sums of its opposite sides are equal (*AB*+*CD*=*AD*+*BC*). Hence we may say that *H*, *K*, *L*, *M* are coincident when *AB*+*CD* =*AD*+*BC*.

Even if our students are working with paper and pencil, they refer explicitly also to the exploration with Cabri ("we have observed"). This confirms the interlacement of exploration and proof. We note that the original property based on the inscribed circle is visual and was obtained with a construction, while the property *AB*+*CD*=*AD*+*BC* is the consequence of a theorem. Thus the final statement is expressed in a form ("…when *AB*+*CD*=*AD*+*BC*") that hides the steps through which students arrived to the statement. Definitely the students are in the synthetic mode of reasoning inside the Euclidean theory.

PHASE 6

Alex and Luca have produced, proved, and stated in a formal way a conjecture. Now they go back to the original problem given by the teacher to look for other results. Again they use Cabri to explore, in a way more systematic than that used initially. As done in the phase 1, they start from the external configuration *ABCD*. This coming back to exploration evidences the cognitive continuity between the phases of exploration, production of conjectures and proof. But they have to stop: the time is over.

FINAL COMMENTS

We summarize the steps of students' reasoning from conjecture to proof:

- *Reading the terms of the problem and translating it in the graphical language*: the role of Cabri is central in interpreting correctly the statement
- *Wandering dragging in search of inspiration for producing a conjecture*: this mode is close to empirical methods used in experimental sciences. This is a moment in which creativity has to be present: Cabri amplifies the students' creativity.
- *"Aha!" moment*: a property is discovered. This provokes a *leap* in the way of dealing with the problem. The mode of using Cabri is reversed: instead of going from the given quadrilateral *ABCD* to the resulting quadrilateral *HKLM*, our students start from *HKLM* and investigate on the facts that may have this quadrilateral as consequence. They apply a method recalling analysis.
- *Dragging with Cabri to search a way for proving the conjecture*: this is a phase in which the students need to reflect. Cabri provides many situations and the students have to find those suitable to their purpose, since, as Poincaré (1899) has observed, it is not enough to produce right situations, you have to choose among all possible situations. The way of thinking here is close to the analytic method.
- *"Aha!" moment*: the students make a construction that inspires the statement of a theorem. Again we feel that Cabri has amplified the students' creativity. Here we have another *leap* in the students' reasoning. The method they follow is mainly synthetic. This leap is marked by the use of paper and pencil instead of Cabri.
- *Inside the Euclidean theory*: the students are able to produce a new theorem through deduction from an Euclidean theorem.

In the strategies applied by our students it is remarkable the presence of methods close to analysis and synthesis, as well as the role of the construction as a pivot between the two methods.

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