CHILDREN'S CONCEPTIONS OF INFINITY OF NUMBERS IN A FIFTH GRADE CLASSROOM DISCUSSION CONTEXT

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"How many are the numbers?", "How many numbers are there between 1 and 2?". These tasks were set in three fifth grade classes in an educational environment characterised by an alternation of classroom discussions orchestrated by the teacher and students' production of individual reports. They allowed us to analyse both the short term evolution of students' conceptions about infinity of numbers through social interaction, and their relationships with the cultural environment. Moreover, during classroom discussions we observed an autonomous shift from the original tasks to a new question "Is it possible that an infinity of numbers exists?". A number of related naive epistemological positions (that seem to depend at least partly on the students' cultural environment) were detected.

INTRODUCTION

Preceding research studies suggest that what children say about infinity depends on both task and context. Both the way of formulating the task (textual aspects) and the mathematical context (e.g. geometrical or numerical) seem to be influential (see Monagham, 2001, p. 240 ; 254). Moreover, students' conceptions of infinity are open to changes and shifts in the short term within a suitable computer learning environment (see Sacristan Rock, 2001). However, we can observe that most research on children's conceptions of infinity takes into account only their individual performances, often in a situation of interaction with an adult (usually, the interviewer); only few research studies deal with classroom social interaction situations. In particular, Bartolini Bussi (1989) describes a teaching experiment on infinity in grade IV based on classroom discussion orchestrated by the teacher. In that situation, some interesting shifts in students' positions were observed and many arguments coming from students' school and everyday life culture entered the debate. The history of mathematics provides some evidence about strong relationships existing in some crucial periods (in ancient Greece and, especially, in the second half of XIXth century) between specific epistemological conflicting positions and theoretical constructions concerning infinity, and general philosophical assumptions depending on the culture of the time (see Jahnke, 2001, pp. 189-192). All this legitimates the following questions:

A) how can classroom discussions (orchestrated by the teacher) influence the short term evolution of students' conceptions about the infinity of numbers?

B) is there a counterpart for young students of what emerges in the history of mathematics, in terms of the relationships between their conceptions about infinity of numbers and their cultural environment?

The aim of the research reported in this paper is to study what fifth students say about the infinity of numbers, taking into account both the educational context (in particular, what happens during classroom discussions orchestrated by the teacher) and the cultural context (in particular, with reference to religion, available technological tools, and school learning about numbers).

THEORETICAL FRAMEWORK

A preliminary methodological problem, highlighted by preceding studies (cf. Monagham, 2001, p. 240), concerns the way of considering what students say in relation with what they think about infinity (and related subjects) within an educational context where relevant hints come from the formulation of tasks and from schoolmates' and teacher's interventions. We decided to use the expression "conception of infinity of numbers" to designate what the student says in a given moment, being aware that in many cases no firm, conscious and clear acquisition, independent of the communication context, underlies his/her utterances. Different analyses of students' conceptions of infinity have been made in preceding studies (see Monagham, 2001 for a survey). We share the point of view expressed by D. Tall (see Tall&Tirosh, 2001, p. 133) about the epistemological distinction (and cognitive tension) between "*'natural' concepts of infinity*," possibly bearing contradictory aspects, *"that arise through extending finite experiences to the infinite case, and 'formal' concepts of infinity framed in modern axiomatic approaches"*. According to the axiomatic method, *"selected finite properties can be formulated to give corresponding axiomatic theories"*. In particular, we will try to detect (in what students say) early traces of a "sequential" point of view (based on the idea of moving from each number to the next one, ... and so forth), which could be related to Peano's axiomatisation of numbers, as well as of a "cardinal" (or "set") point of view, based on the consideration of "quantity of elements", which could be related to a systematisation of numbers based on set theory.

Focus on the relationships between students' conceptions and their cultural environment drew our attention on the need of considering arguments that enter a discussion about infinity of numbers and come from non-numeric domains. Different tools were available; we have considered "Mental models" (see Fischbein, 2001) and "Conceptual metaphors" (see Nunez, 2000). In particular, Nunez provides a frame to describe mental operations performed through the use of conceptual metaphors (defined as *"cross-domain mappings that project the inferential structure of a source domain onto a target domain")*. It allows also to distinguish between different kinds of conceptual metaphors that intervene in our study: mainly grounding metaphors (the source domain is in everyday life experience); and linking metaphors (the source domain is mathematical but different from the domain of numbers under scrutiny).

Considering the evolution of students' conceptions from a social construction point of view, the need for analysing such evolution in relationship with the educational setting (classroom discussions, etc.) and the cultural environment led us to adopt a Vygotskian perspective. This choice was made in order to frame and interpret in a coherent way some specific phenomena related to social interaction and cultural belonging (in particular, the internalisation process; the shift from content conceptions to meta-theoretical considerations) – cf Vygotskij (1990)

We conclude this Section with two assumptions concerning terminology taken from Boero, Douek & Ferrari (2002): *argumentation*: "*both the process that produces a logically connected (but not necessarily deductive) discourse about a subject, as well as the text produced by that process*"(the context will allow to choose the appropriate meaning*). Argument*: "*a reason or reasons offered for or against a proposition, opinion or measure* (Webster Dictionary) – *it may include linguistic arguments, numerical data, drawings, and so forth*".

THE TEACHING EXPERIMENT

Three fifth-grade classes of 16, 21, 22 students were involved in the study in March 2002. Teachers belong to the Genoa Research Group in Mathematics Education.

The sequence of tasks and the educational context

According to the usual sequence of activities in the Genoa Group Project, classroom activities concerning infinity of numbers were organised as follows:

FIRST PHASE: First classroom discussion (20'-25'): *"Sometimes in the last months we have met the problem of how many are the numbers –whether finite or infinite. Now it is the moment to start dealing with this problem in depth".* The discussion was followed by the production of an individual text concerning *"What do you think about the problem that we have discussed?".* The following day, 3 or 4 (according to the class) individual texts, representing different positions, were selected by the teacher, photocopied and distributed to all students, with the task of identifying both analogies and differences with personal productions. The teacher helped some students (through 1-1 interaction) to perform this task. The aim of this phase was to create a shared baggage of ideas and references for the subsequent phase.

SECOND PHASE (1-2 days later): Second classroom discussion (35'-40') *"In order to go in depth into the problem that we have discussed in the last few days, it is useful to try to answer the following question: 'How many numbers are there between 1 and 2?'. Try to do your best to answer this question!".* Then students were asked to produce an individual, written *"detailed report about your position and its motivations".* The choice of setting a task different from *"How many are the numbers?"* was meant to avoid mere repetitions of arguments and positions that had been expressed in the preceding phase, and engage students (who had not produced some arguments) to appropriate them in order to tackle a different question.

Concerning the mathematical background, most students were able to deal with finite-decimal numbers. They were able: to perform arithmetic operations with them (both using paper-and-pencil methods, and pocket calculators); and to use them in measuring activities, in particular to represent/read numbers like 2.5 or 0.82 on the ruler. They had already experienced the fact that the division 1:3 produces the result 0.3333...

Criteria for analysing students' productions

We have tried to analyse the evolution of students' conceptions during the discussions, and in comparison with their individual texts, by considering the "sequential" and "cardinal" points of view (cf our theoretical framework). The progressive maturation and clarification of such conceptions were also analysed. In order to study the relationships between students' conceptions of the infinity of numbers and their cultural environment, we have considered both the nature and the sources (technology, school subjects, religion, and personal preceding elaboration) of the arguments brought by students (specially metaphors). The origin (when, why and how) and the distribution and possible evolution of the most frequent arguments in the three classrooms were considered too, in order to detect some effects of classroom discussions on the students' argumentation.

SOME OUTCOMES

As expected according to preceding studies (cf. Monaghan, 2001 for a survey) different positions concerning the answers to the main tasks *("how many are the numbers"? "How many numbers are there between 1 and 2 ?")* were expressed initially; both the "sequential" and the "cardinal" conceptions emerged in each class in the motivations for each position reported in the written texts. Some language ambiguities were detected (cf Monaghan, 2001, pp. 240, 241) : in particular the sentence*: « I cannot count all the numbers »* might mean *"I have not enough time to count all the numbers, they are too many"* or *"I cannot reach the last number".* Another ambiguity depended on the use of *"infinito" (in Italian)* as a noun *("infinity")* and an adjective *("infinite").* And the adjective *"infinito"* was applied both to a number like 1.1111... and to the sequence of natural numbers: 1, 2, 3...

Consistently with the aim of this study, we will analyse: how positions (and conceptions related to them) evolved in the educational setting described in Subsection 3.1; how language ambiguities were dealt with; and what links with the cultural environment emerged.

The evolution of students' conceptions

21 students out of 59 clearly moved from one position to another (16 students moved from the "finite" position to the "infinite" position - see Debora below; 5 in the opposite direction – see Sabrina, in the last Subsection), by taking into account different positions - and the related arguments - brought into the debate by their schoolmates. For instance, Debora initially said that only ten numbers exist between 1 and 2 *("2.1; 2.2; ...2.9")*, then (exploiting a hint from a schoolmate) she considered also 2.01; 2.02; etc. up to 2.99 *("still a finite number"*), then she said that *"in any case, they are finite: let they be 9 or 99 or 999, it is the same! A finite number").* A critical point was reached when Ivan proposed the sequence 1.1; 1.11; 1.111; etc. (Debora): *"now I understand: I cannot get the last number, I can go on in an endless counting"*.

As concerns students' conceptions, in some cases we have observed that students moved from a "cardinal" view to a "sequential" view (see Debora), in other cases, they moved in the opposite direction. In general, classroom discussions allowed to develop more and more precise and sophisticated positions (through selection and integration of arguments brought by peers, and/or refinement of arguments, under the pressure of contrasting peers: see Debora again). An interesting aspect was the repeated crossing of lines of argumentation where the "sequential" view and the "cardinal" view became more and more sophisticated as a consequence of the need to consider the others' views (see Paolo's interventions in the following excerpt) :

> (Paolo): *"There is no last number, if I make 1.9; 1.99; 1.999 I get nearer and nearer to 2 but there is always another number in between, like 1.9999"*

(Beatrice) *"You say that they are so many, but all of them cannot stay there, they are too many!"*
(Paolo) *"but they are as many as… as if I count 1, 2, 3, 4, as many as 9s, but with whole numbers I go on with steps of 1, and I carry on without finding one last whole number, while here I go on with a smaller and smaller quantity*

Some students' positions were refined through an appropriate treatment of language ambiguities during discussions. The ambiguity inherent in *"the impossibility to count all numbers"* was clarified in the three classes through the teachers' hint of considering the numbers of the pocket calculator. Students realized that *"there is a last number ; they are finite, but we cannot count all of them because we have not enough time, while in the case of integer numbers there is no last number".*

Classroom discussions were a source for social construction of knowledge and subsequent internalisation. Here is an example where some arguments produced by peers were integrated (in the subsequent individual text) as components of an inner dialogue, echoing the classroom debate, that brought to a substantial evolution of the student's position: (still Debora, second text) *"If I think that numbers between 1 and 2 are finite, be they 9, like 1.1 up to 1.9; or 99, like 1.01 up to 1.99, I cannot cope with the example brought by Ivan. Indeed the sequence 1.1; 1.11; 1.111; 1.1111 is endless, and those numbers do stay between 1 and 2. To answer the remark made by Sabrina, I can observe that if she would say that one point one million times 1 is the longest number for her mind, I could say: one point one million and one times1, and I am sure that she could think about it as well! So, the sequence is really endless".*

Metaphors

Students' conceptions about infinity of numbers between 1 and 2 were related by them, during the discussions, to arguments belonging to different cultural domains. We can distinguish between three kinds of metaphors:

- metaphors where the source domain was mathematical - but in a different domain of mathematics, generally geometry: (Federica): *"The points on the line between 1 and 2 are finite, because they cover only a short line, and the same should happen for the numbers between 1 and 2"*

- metaphors related to ordinary life experience, like in the following excerpt:

(Valentina) *What does it mean to say that infinite numbers exist, if we cannot count them because we must die?"*
(Stefano*) "I agree, man is not everlasting but life is everlasting"*
(Valeria*)"The woman's body ends, but she creates another woman, and so life goes on to infinity"*
(Emanuele) *"Numbers create other numbers, to infinity, by multiplying. Each number is finite, but an infinite list is produced".*

- metaphors directly or indirectly related to religious ideas (the eternity of God and/or soul, the unboundedness of Universe as the realm of God): (Emi) *"Numbers are infinite, because the time needed to count them would be infinite, like the life of our soul. Infinite numbers do exist, because it is like our soul: we cannot reach its end, because we die before, but this fact does not mean that soul gets to an end".*

What is the role of these metaphors in classroom discussions? In few cases their use seems to depend on the mere need of communication in order to overcome the lack of technical terms. In some cases, they are produced (or used by someone who did not produce them) as arguments for the plausibility of a hypothesis (see the example of Federica). In other cases, they support the shift to meta-theoretical positions, like in the preceding case of Emi.

The problem of existence of infinity

Preceding considerations bring us to the most interesting (in our opinion) result coming from this study: the emergence in the three classes of an "existence problem" for infinity as a relevant issue for students to deal with when they must choose between accepting or rejecting the idea of an infinity of numbers. The existence problem emerged (in different modalities) as a result of the transition from the debate about *"how many numbers..."* to the question *"can infinite numbers exist?..."*. Here is an example about how this transition occurred through the intermediate problem of the existence of numbers that cannot be reached by counting:

> (Clelia) *"Numbers are infinite, because we can imagine numbers that grow bigger and bigger, with no limit",*
> (Stella) *"And also smaller and smaller, like 0,1; 0,01; 0,001"*
> (Enrico) *"We can be sure that a number exists only if we can reach it by counting*"
> (Amelia) *"I cannot count up to one million, but one million exists: we used it for liras"*
> (Ezio) *"I agree with you for one million and for one billion too, but for numbers that we do not know... That we do not use... Do they exist?"*
> (Sabrina) *"If we cannot touch or see something, we are not sure that it exists"*
> (Clelia) *"But we think that God exists! And God is eternal! Like the time necessary to count all numbers".*

Students dealt with the existence problem under three different perspectives:

- existence considered as the possibility of "experiencing it": this test usually brought students who chose it to the exclusion of the possibility of infinity of numbers. In some cases this position took the flavour of a proto-philosophical assumption, in other cases it heavily relied on pragmatic considerations (related either to paper and pencil ordinary uses of numbers, or to the availability of calculators). The case of Sabrina is interesting: she moved from a mature expression of potential infinity existence *("if I write 1, 1,1, I get an infinity of numbers, because I can add how many 1s I want. There is not a last number, I could increase it by adding 1 to the right!",* to a doubt preparing rejection, and related to accessibility to experience and pragmatism *("I cannot say if there exists a last number: it may be that we cannot think about a longer number, it could not enter our mind. And in any case a longer number would be a useless number, if we cannot think about it."* But Fadel considered the impossibility of experiencing infinity as a necessary condition for its existence: *"We cannot arrive to the last number, so numbers are beyond our possibility of knowing. Infinity cannot be known, because if we could get to know it, it would not be infinite."*

- existence thought of as an inner consequence of the structure of the number system: this was frequently related to the "sequential" point of view about infinity *('by adding 1 we always get a bigger number , but we cannot reach the last number'*, *'by writing 1 on the left we still get a number between 1 and 2, different from the preceding one: 1; 1.1; 1.11; and so on'*). But in one case the same consideration brought to the exclusion of the possibility of infinity (due the impossibility of reaching 2 on the line through the points 1.9; 1.99; 1.999 - two conflicting tacit models related to infinity seem to intervene: cf Fischbein, 2001).

- existence as the possibility of an independent, non-accessible reality frequently related to religious transcendence and/or space and time unboundedness: *"Numbers do exist, and always existed, and always will exist even if we do not think about them. They are endless. They are like God, who already existed before the creation of man, and man was not there to think about Him"*

Comparing the three classes, we have observed that in one of them the discussion about the "existence problem" did not develop very well because of frequent interventions of some students who refused to consider the infinity of space, or the eternity of God, as pertinent arguments in a discussion about the infinity of numbers.

DISCUSSION

Our study suggests that the complexity of the problem of children's conceptions of the infinity of numbers in school is perhaps even bigger than preceding studies had revealed. We must deal with this problem not only in individual, developmental terms (see Fischbein, 1979; Monagham; 2001) and in the social construction perspective (see Bartolini Bussi, 1989) ; we also need to consider the cultural environment and the didactical contract (specially as concerns arguments that are legitimate for students).

As concerns the "existence problem", we can ask ourselves what were the conditions that enabled students to pose it and deal with it.

The classroom discussion context seems to be a convenient environment for this because each position forces the supporters of the other positions to move to a meta-theoretical consideration, in order to defend their own position. This happened in the three classes and fits what was observed in some teaching experiments concerning other subjects and conceived in a Vygotskian perspective (see Bartolini Bussi&al, 1989: shift from empirical to theoretical, and then to meta-theoretical considerations about geometrical entities). The analysis of the data collected in our teaching experiment (see the previous Subsection) suggests that, in a classroom discussion context, the emergence of the "existence problem" and the possibility of a passionate debate about it might depend on:

- Preceding classroom experience of exploration of the number domain (specially as concerns the generation of decimal numbers like 0.33333...); and preceding discussions about the "numbers of the pocket calculator".

- Frequent questioning about what is beyond the immediate reach of our experience, nurtured in other classroom activities (production of interpretations, of predictions, etc.: see Douek, 1999 for some examples).

- The familiarity with an “existence problem” related to transcendent realities (like the soul, or God) due to parallel Catechism. This could explain the frequent shifts to the consideration of arguments taken from religion.

As concerns educational implications, the emergence of an “existence problem” for mathematical entities related to the question “How many numbers ...” suggests to consider the problematique of the infinity of numbers as an extremely interesting opportunity to develop the students’ sensibility and (later) their awareness about the nature of mathematical entities. Concerning this issue, we must recognise that the students’ potential of epistemological thinking is surprisingly high in grade V!

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