TOPIC GROUP 4

ARGUMENTATION AND PROOF

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Topic Group 4 of CERME4 on "Argumentation and Proof " had 15 participants from different countries across Europe. During its sessions, no formal presentations of the 9 papers were made, but every author had a chance to present her/his main ideas. The discussion was organized around three main themes that emerged from the papers prepared for the Topic Group. This report is organized likewise, in that it presents an account of the arguments and comments made in the group in terms of the three themes rather than in the order in which the discussion occurred.
The three themes were

- The meaning of proof in mathematics education
- Comparing the teaching of proof at school
- Problem solving and Conjecturing.

Theme 1: The meaning of proof in mathematics education

The discussion on "The meaning of proof in mathematics education" was opened by David A. Reid who presented the working group with examples of proofs that raised the central question in his paper, "Is there any prototype of proof?". In line with the analysis in the paper, participants generally rejected certain arguments as mathematical proofs and accepted others, even though they were generally not able to define the characteristics of a prototype proof. Thus it seemed that when it came to judge arguments as proofs or non-proofs there was common ground, regardless of the different epistemological positions that participants might have held.

Whether there was common ground concerning teaching approaches towards proof is less clear. It is an interesting question, though, whether researchers' epistemologies have a more significant influence on the teaching approaches they espouse. This still needs further investigation. The “genetic approach to proof” presented by Hans Niels Jahnke clearly is an example of a coherent position in this respect. Starting from a specific and explicitly stated epistemological position, “to treat geometry in an introductory period as an empirical theory”, Jahnke expounds a teaching approach that evolves from this epistemological position. First he points out that viewing mathematics introduced to novices as an empirical theory provides an explanation for the strong attraction to experimental verification felt by students when a conclusion is new and important to them. Second, Jahnke insists that therefore the meaning of proof should not be introduced as a replacement or contrast to measurement. Instead measurement and proofs should be combined into a “small theory”. Jahnke’s theory provides a nice explanation for a phenomenon reported by Küchemann and Hoyles. They found that only a small minority of their sample of high attaining school students could give a structural explanation for the divisibility of, say, 100! by 31, but even students in this group wanted their argument confirmed empirically. The relationship between empirical evidence and proof gains a new meaning in the “genetic teaching approach” that is consistent with the background epistemological position explicated by Jahnke. Jahnke’s notion that there is a kind of dialectic or mutual support between intellectual proofs (to use a term of Balacheff’s (1999), to identify the case when arguments are detached from action and experience) and recourse to data (pragmatic proofs) throws a new and refreshing light on their relationship and on the seemingly negative findings of Fischbein and Kedem (1982) and Vinner (1983)

Viviane Durand-Guerrier’s contribution suggests that her epistemological position is not likely to be based on the same beliefs and principles as Jahnke’s. Though she does not explicitly label the main ideas in her paper as an epistemological position, many of them can be read as such. As with Jahnke, she makes us aware that we need stronger theoretical backgrounds in order to better understand students’ thinking and difficulties. However, the theoretical tools she refers to are distinctively different from the theoretical approach of Jahnke. Following Quine and others, Durand-Guerrier claims that predicate calculus, in particular “natural deduction” is a powerful tool to analyse proofs “in a didactic perspective”. According to Durand-Guerrier logic can help us to understand students’ difficulties with proofs, as a logical analysis makes us aware of the various properties of objects and their various nature that is silenced so often. Durand-Guerrier emphasizes that logic should also play a role in the teaching of proof, in the sense that predicate logic “provides an extension of the classical tools used in didactic of mathematics to analyse proofs”. However, in her paper she is not propounding the idea that logic should be taught to students, neither is she making any hypothesis about logic being a powerful tool for students to understand proofs. Instead, her focus is on mathematics educators, teachers as well as researchers.

Theme 2: Comparing the teaching of proof at school

Despite the apparent similarities, the teaching of proof presents a great variability across different countries. Not many studies have been devoted to the comparison of the teaching of proof; the study carried out by Knipping (CERME3) and presented at CERME3, was based on classroom observation.. A different perspective is taken by Richard Cabassut who accomplished his analysis through the comparison of textbooks. In his paper, Cabassut uses examples of proofs found in French and German textbooks to analyse underlying differences in proof concepts and teaching approaches. His cross-cultural comparisons and analyses indicate that mathematical properties and concepts anchored in the curriculum can be one reason why specific proofs occur in textbooks of one country but not of another. But Cabassut also suggests that this is not the whole story, as certain mathematical properties are not explicitly referred to in a textbook’s proofs, though the properties are explicitly mentioned in the corresponding curriculum. Therefore other factors must have motivated authors of school textbooks in their choice of proofs. However, we may still be some distance away from finding the tools to decipher those factors. For example, it seems that different functions of proofs, as a category for analysing textbook proofs, do not reveal underlying cultural differences in teaching approaches, at least comparing German and French text books. Also Cabassut's analysis of arguments, based on the Toulmin model, does not obviously uncover cultural differences in proof concepts. Two divergent conclusions can be drawn from this. i) There are no significant cultural differences between proofs in German and French textbooks. This conclusion would suggest that word-for-word translations of textbooks should be usable without any problem in the other country. Those who have taught in French or German schools might doubt this. ii) There are cultural differences, but to reveal them remains a challenge. International groups such as those meeting at CERME provide a possible avenue to a better understanding of cultural differences in mathematics education.

Kirsti Nordström in her contribution to the working group also takes a comparative approach to textbook analyses, but from a very different perspective. Her motivation for these analyses resides in the learner's perspective and the fact that textbooks and teachers guides influence the practices and choices teachers make in the classroom. Nordström rises the important question of how proof items in textbooks "relate to students' access to proofs". By examining two series of commonly used Swedish upper secondary school books, she found that different definitions for proof occurred in the same textbook and that the notion of proof was often left implicit or not defined in a mathematically appropriate and meaningful way. This is likely to lead to substantial difficulties for teachers and students and, as Nordström reminds us, the important question remains: "By what means is it possible to make proof and its different aspects visible to students at the upper-secondary level?"

The longitudinal study undertaken by Hoyles and Küchemann compares not across cultures and textbooks, but students' responses over time. Their large-scale survey using written questionnaires to investigate students' understanding was supplemented by case studies to identify teaching practices, which may have influenced these responses.

These various studies and the resulting group discussion underline the importance of using a variety of methods in our research to overcome the limitations of any single method, and the importance of considering teaching practices more globally, not just taking a single perspective, be it student results, curriculum documents or text books.

Theme 3: Problems and Conjecturing

Students' approaches to mathematical problems and students' production and validation of conjectures were discussed in the light of three papers contributed by members of the working group. Lourdes Figueiras & Jordi Deulofeu presented observations and analyses of first year university students' conjectures for Heron's Problem. Oleksiy Yevdokimov described the conjecturing processes of secondary school students about properties of special lines in a triangle. Consuelo Cañadas Santiago & Encarnación Castro Martínez presented an analysis of secondary students' (inductive) reasoning in the context of the problem of adding two even numbers.

One suggestion put forward in the ensuing discussion was that slight changes in the formulation of a task could radically alter the way it was solved and consequently the way students prove the correctness of their solution. For example, Figueiras & Deulofeu had posed Heron's Problem (in their university course for future primary student teachers) as follows: "Let s be a line and A and B two points at the same side of s. For which point P in s is AP+PB the shortest way joining A and B?". Knipping had exposed a similar group of students at her university to Herons's Problem in a slightly different way. In her formulation it was not assumed that A and B were points at the same side of the line s, but only that they were not *on* the line s. This slight difference seems to have mattered here: in particular, it allowed students to make productive use of their solution of the case where A and B are on opposite sides of line s, to solve the more elusive standard case. On the other hand, in a subsequent investigation by Küchemann, using wording similar to that of Figueiras, he found that exhorting students to work dynamically by inserting the sentence "Explore what happens to the distance AP + BP as P moves on S" seemed to make no difference to their approaches.

Comparative analyses like this may make it possible to understand students' conjectures in the context of how tasks and problems are formulated and posed to students. It may also lead to a better understanding of the interrelation of tasks, conjectures and students' engagement in mathematical reasoning. Yevdokimov's work with open tasks such as "Find out as many properties of a bisector of a triangle as you can", lead to an interesting working group discussion. Not all students were in every case able to prove the conjectures they came up with. This suggests that even if students find the process of conjecturing to be fruitful and satisfying they might still be discouraged by the task of proving. This runs counter to the commonly held hypothesis that the production of conjectures motivates students to want to construct proofs. Thus we have to be aware of the potential of tasks not only for producing conjectures, but also in respect of students' aptitude for proving (or refuting) their conjectures.

This problématique is also provoked by the paper of Cañadas Santiago & Castro Martínez. The authors had observed students who engaged in justifying the result obtained when adding two even numbers and found that students would use "inductive reasoning" to justify their conjecture. Again, the difficult part for the students was not coming up with a conjecture, even for the general case, but in deriving a generally true justification. Promising research on this problématique has been done by some of our Italian colleagues who have introduced the notion of "cognitive unity". This aims to describe the complex relationship between the process of producing a conjecture and proving it. A paper on the notion of cognitive unity was presented at CERME3 (Pedemonte, 2003). Applying the notion to the relatively simple task used by Santiago & Martínez suggests that if students form a conjecture on the basis of empirical evidence only, then they will try to construct a proof on the same basis. Küchemann's current work with teachers, using similar simple tasks, suggests that in part students' difficulties may be a matter of initiation In fact adopting deductive, as opposed to empirical, approaches may not be so natural as it could be expected and seems to require a specific didactic intervention.

As a complement to the notion of 'proof prototype' discussed in Theme 1, it is sometimes useful to think of 'task prototypes'. Thus for example, Heron's task in the form presented by Figuieras would seem to be especially useful for generating conjectures that turn out to be false (and hence need to be refuted). It is also a task where some students might fruitfully draw on experiences in physics (angles of incidence and reflection, or the notion of a weight supported by a string attached to the focii of an ellipse...). Consuelo Cañadas spoke of the classic task, "If n straight lines all intersect each other (inside a circle), how many regions do they make (inside the circle)?". It is difficult (impossible?) to solve this task generically, i.e. simply by looking at one (general) case. On the other hand the task can be approached ‘incrementally’ by looking at what happens when another line is added (the nth line produces 1+n–1 = n new regions), but this is relatively complex. By contrast, consider another classic task, “If n straight lines all intersect each other, how many points of intersection are there?”. This can be solved generically (each line intersects the n-1 other lines, but this is counting each intersection twice as it is formed by two lines, so there are n(n–1)/2 points of intersection). Of course, as with the previous task, it can also be tackled incrementally or by induction from specific cases.

Küchemann has recently been using the following task in secondary schools: “Find the sum of three consecutive numbers. What do you notice? Prove your result.” Students tend to notice many different things and tend to work empirically. Thus if they notice that the sum is 3 times the middle number, they tend to confirm this by means of examples. Of course, the task can also be approached structurally, along the lines of ‘The first number is 1 less than the middle number and the third number is 1 more than the middle number; hence the sum is equivalent to 3 times the middle number”. Küchemann would argue that students would benefit from becoming familiar with arguments of this sort, which are often not difficult to understand. However, not all tasks can be tackled in this way. Take this example: “Find the product of three consecutive numbers. When is the result a multiple of 24? Prove your finding.”. Here it is fairly straightforward to find out and prove that the condition is satisfied when the first number is even; however, it is harder to realise that it is also satisfied when the middle number is a multiple of 8. Thus, in the school context at least, this a good example of a task where students might benefit from using an empirical approach, and indeed from generating lots of data, for example by using a spreadsheet. The argument here is that teachers and students would both benefit from not only knowing that there are different ways of solving problems and constructing proofs but that different tasks lend themselves to different approaches.

As a final remark rising from the group discussion we would like to stress the importance of grounding the notion of proof in problem solving context, that means developing a culture of arguments supporting one’s own solution through the systematic use of open problems, where producing a conjecture accompanied by the arguments supporting it. Tasks design becomes crucial and as clearly emerged from the discussion both the individual and the social component of the proving process.

Generally speaking, it becomes interesting to identify contexts where mathematical knowledge might emerge from solving problems and validating solutions, according to different school levels and different curriculum requirements.

Never the less, overcoming the possible gap between arguments supporting a statements (for instance, empirical evidence based on a limited number of example, or authoritative reasons) and a mathematical proof (i.e. arguments acceptable in the mathematics community) remains a major difficulty requiring the specific intervention of the teacher, who as a cultural mediator introduces students to mathematics practices, and among others to proving.

Although any contribution to the working group directly addressed the issue of teachers training, the key role recognized to the teacher - both in designing of a problem solving environment and in guiding the evolution towards shared mathematically acceptable arguments – asks further investigation in this field.

References

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List of the contribution

Theme1

- David A Reid

The meaning of proof in mathematics education

- Hans Niels Jahnke

A genetic approach to proof

- Viviane Durand-Guerrier

Natural deduction in Predicate Calculus A tool for analysing proof in a didactic perspective

Theme 2

- Richard Cabassut

Argumentation and proof in examples taken from French and German textbooks

- Kirsti Nordström & Clas Löfwall

Proof in Swedish upper secondary school mathematics textbooks - the issue of transparency

- Dietmar Küchemann &Celia Hoyles

Pupils' awareness of structure on two number/algebra questions

Theme 3

- Lourdes Figueiras & Jordi Deulofeu

Visualising and Conjecturing Solutions for Heron's Problem

- Oleksiy Yevdokimov

About a constructivist approach for stimulating students' thinking to produce conjectures and their proving in active learning of geometry

- Consuelo Cañadas Santiago & Encarnación Castro Martínez

Inductive reasoning in the justification of the result of adding two even numbers