ABOUT A CONSTRUCTIVIST APPROACH FOR STIMULATING STUDENTS' THINKING TO PRODUCE CONJECTURES AND THEIR PROVING IN ACTIVE LEARNING OF GEOMETRY

Oleksiy Yevdokimov

Kharkov State Pedagogical University, Ukraine

The paper describes processes that might lead secondary school students to produce conjectures in a plane geometry. It highlights relationship between conjecturing and proving. The author attempts to construct a teaching-learning environment proposing activities of observation and exploration of key concepts in geometry favouring the production of conjectures and providing motivation for the successive phase of validation, through refutations and proofs. Supporting didactic materials are built up in a way to introduce production of conjectures as a meaningful activity to students.

Background and theoretical framework

Conjecturing and proving activities have traditionally been given significant attention throughout the Grades in the schools. In the most of the countries current mathematics education standards contain a special section on reasoning and proof, investigations, conjectures, evaluation of arguments and the use of various methods of proofs (e.g. NCTM Standards, 2000; MIUR, 2004; UMES, 2003). Therefore, different questions concerning argumentation and proof are constantly in the focus of researchers and mathematics educators' interest.

During the last years a considerable amount of research, concerning the processes of conjectures production and construction of proofs was connected with interactive learning environment and different software (Harel and Papert, 1990; Davis, 1991), in particular, dynamic geometry software (Arzarello et al, 1998; Furinghetti and Paola, 2003; Mariotti, 2000). Neither considering a technology aspect within the theme of argumentation and proof in full nor restricting the use of technology in that direction, we would like to put into consideration a constructivist didactical approach, which, on the one hand, can be successfully used in stimulation students' active research interest while learning mathematics, in particular, geometry, on the other hand, provides possibility for investigation students' abilities in their development to produce conjectures.

Within the long-term research project on active learning of mathematics in a constructivist framework for secondary school students the author gave a course on plane geometry for 45 students (8^{th} Grade, 14-15 years old) during the year. All students didn't take any courses before, which were aimed on formation of skills to produce conjectures specially. Teaching programme of the course consisted of 16 theoretical modules with support of the didactical materials for each of them.

The aim of the paper is to describe a didactical approach to organise students' thinking leading to conjectures production on the base of materials of this long-term research project, which were worked out and used as support for active learning of geometry in a constructivist framework. These materials were built up in a way to introduce conjectures production as a meaningful activity to students.

Our theoretical position is grounded in the theory of active learning processes in mathematics (Hiebert, 1992; Wang, Haertel and Walberg, 1993). We took into account that current learning perspectives incorporate three important assumptions (Anthony, 1996):

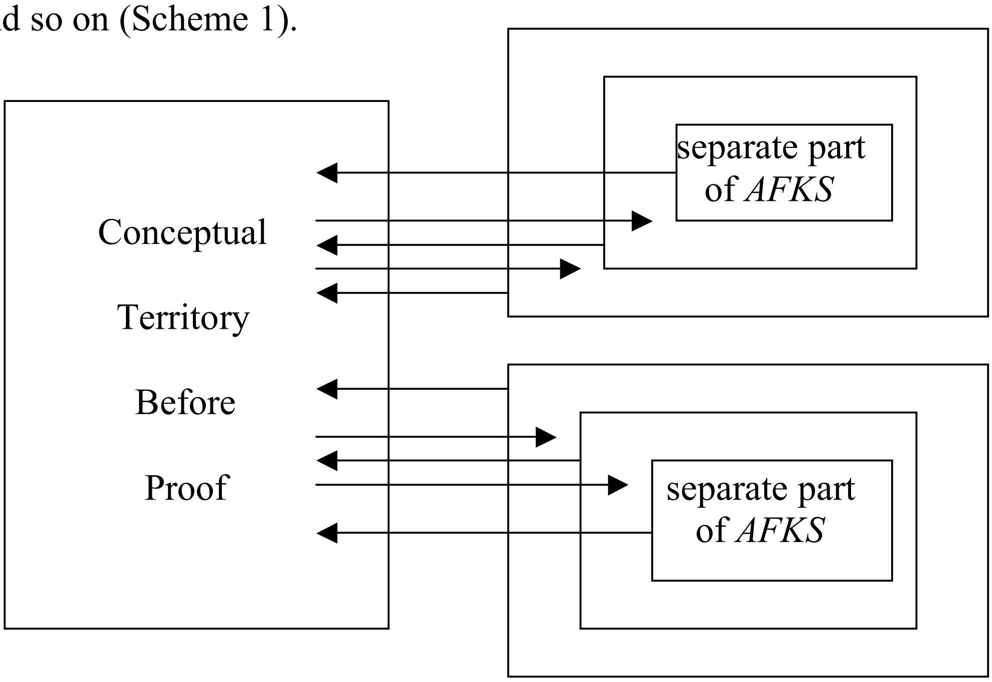
- learning is a process of knowledge construction, not of knowledge recording or absorption;
- learning is knowledge-dependent; people use current knowledge to construct new knowledge;
- the learner is aware of the processes of cognition and can control and regulate them.

From a constructivist perspective it is easier for a student, under appropriate arrangement of teaching, to act as an architect, to reveal the truth and construct new knowledge, than to learn ready-made knowledge without understanding its origin, meaning and interrelations (Davis, 1991). In other words, *"learning is a process of construction in which the students themselves have to be the primary actors"* (von Glasersfeld, 1995). Thus, the view of the learner has changed from that of a passive recipient of knowledge to that of an active constructor of knowledge.

We would like to consider conjecturing process from the didactical point of view, i.e. to show how it can be constructed similarly to the process of mathematical research and how this kind of teaching contributes to the development of students' mathematical thinking. de Villiers (1996) and Hanna (2000) noted that in actual mathematical research mathematicians have to first convince themselves that a mathematical statement is true and then move to a formal proof. At the same time conjecturing with verification, exploration and explanation constitute the necessary elements that precede formal proof (*"conceptual territory before proof"*, Edwards, 1997). Yevdokimov (2003) proposed to consider an *Active Fund of Knowledge of a Student* (*AFKS*) in the given area of mathematics. As *AFKS* we called student's understanding of definitions and properties for some mathematical objects of that domain and skills to use that knowledge. The key question in the context of conjecturing and proving was the following: where, when and for what of mathematical objects a student would apply a certain mathematical property for proving and whether it would be necessary to apply that property generally in that case.

Modifying this idea, we distinguished the separate (smaller) parts of *AFKS*, which deal with a certain mathematical object, for consideration and, after that, motivated student's using that separate parts of *AFKS* altogether as a whole *AFKS* for production new conjectures and constructing their proofs, even for other mathematical objects. Therefore, on the first stage of our research we attracted students' attention and focused exploratory work on using and developing the separate parts of their *AFKS* as much as possible. On the second stage, we started to increase *AFKS* properly, stimulating students for active mental work with using all separate parts of *AFKS* simultaneously.

We would like to emphasize a dual correspondence between separate parts of *AFKS* (personal attribute) and “conceptual territory before proof” (impersonal attribute). More precisely, parts of *AFKS* act in “conceptual territory before proof” and vice versa, Edwards’ term contributes to increasing every part of *AFKS*, which acts in it again and so on (Scheme 1).



Scheme 1.

Thus, on the first stage of the research within the every module the main task for developing and increasing the separate parts of *AFKS*, was the following:

“Find out as many properties of a certain mathematical object as you can”.

On the second stage of the research within the every module we proposed for students the following task:

“Find out properties of something using your previous findings for certain mathematical objects”.

Now we need to clarify what it means: to find out or reveal a property. On the one hand, any property gives relationship between conjecturing and proving, since for obtaining a property we have to produce conjecture and then prove it. On the other hand, any property gives a clear structure from premise to result (Scheme 2). And such form of any property is a final product of students’ findings for the task.

$$\boxed{\text{Premise}} \Rightarrow \boxed{\text{Result}}$$

Scheme 2.

At the same time, in generalising context a property can be considered in three more forms, which highlight importance of all-sided investigation in relationship between conjecturing and proving (Scheme 3).

$$\boxed{\text{Premise}} \Rightarrow \boxed{?}$$

$$\boxed{?} \Rightarrow \boxed{\text{Result}}$$

$$\boxed{?} \Rightarrow \boxed{?}$$

Scheme 3.

However, a mathematical conjecture does appear from nowhere, without links to its *"historical-mathematical neighbourhood"* (Yevdokimov, 2004). Conjecture is a result of applying *AFKS* to different mathematical objects within *"conceptual territory before proof"* taking into account possible generalisation, systematisation and analogue. Like Brown and Walter (1990) we propose to consider *"situation"*, an issue, which is a localised area of inquiry activities with features that can be taken as given or challenged and modified.

Turning to the forms, it is necessary to note that for finding any property students have to move through its forms of Scheme 3 for obtaining that property in the form of Scheme 2. Moreover, the first two forms of Scheme 3 refer to the first stage of the research and the last form refers to the second stage. Of course, it doesn't relate to all properties, which were known for students before. Here, we speak of the development of conjecturing process in the student's mind only.

For illustration the schemes above we would like to demonstrate one well-known property of bisectors of a triangle in all forms (property B_1 from Appendix). It is shown in the following table.

<table>
<tr><td colspan="2">Property</td><td colspan="3">All bisectors of a triangle intersect in one and the same point</td></tr>
<tr><td colspan="2">Scheme 2</td><td>We have a triangle and three its bisectors</td><td>⇒</td><td>Bisectors intersect in one and the same point</td></tr>
<tr><td rowspan="3">Scheme 3</td><td rowspan="2">Property in the generalising context</td><td>We have a triangle and three its bisectors</td><td>⇒</td><td>?</td></tr>
<tr><td>?</td><td>⇒</td><td>We have a point inside a triangle</td></tr>
<tr><td>"Situation": a triangle and its components</td><td>?</td><td>⇒</td><td>?</td></tr>
</table>

Table 1.

Description of students' activities and data collection

Taking to attention the scope of the paper we would like to describe and characterise students' conjecturing and proving activities for a part of the module concerning a triangle and some of its properties and present a sample of the didactical material (see Appendix) for this part.

We proposed three tasks in succession for students' work on their own. Sufficient period of time was given for every task. Our first task was the following:

<u>Task 1.</u>

"Find out as many properties of a bisector of a triangle as you can".

We would like to note that some bisector's properties were connected with height and median of a triangle. Therefore, in the first part students had been asked to explore the similar tasks for height and median too:

<u>Task 2.</u>

"Find out as many properties of a height of a triangle as you can".

<u>Task 3.</u>

"Find out as many properties of a median of a triangle as you can".

As we mentioned above such tasks were aimed on developing and increasing separate parts of AFKS ($AFKS_{bisector,}$, $AFKS_{height,}$, $AFKS_{median}$ correspondingly).

It is necessary to note that some properties were known for students before, therefore, at first, we asked students to point out all known properties. After that, students could begin their inquiry work to produce conjectures and prove them. At the same time students had been asked to prove every property without any dependence whether it was indicated as known before or proposed as a conjecture. If a student proposed at least 5 conjectures independently, at least 3 of them with proofs, and was not able to produce other conjectures, then he/she had access to didactical material with extended list of properties for bisector (see Appendix), height and median. After that students had been asked to prove those properties, which were unrevealed or unproved for them before. At these stages help of a teacher was an acceptable, but not necessary condition for students. Instead of teacher's help students could take advantage of extended didactical material (it is omitted in the paper), which contained not only properties themselves, but short remarks and instructions for proving every property. After getting acquaintance with proofs students took for consideration the next task.

This procedure was repeated in succession three times for every task. Results became better every time, it witnessed that most of the students increased their AFKS and developed their abilities to produce conjecture. Full data collection of students' progress in increasing their $AFKS_{bisector}$ in the first stage of the research is given in Table 2. The same tables were formed for $AFKS_{height,}$ and $AFKS_{median}$ correspondingly (they are omitted in the paper).

Property	B_1			B_2			B_3		
Status of the property	was known before	proposed as a conjecture	proof was given	was known before	proposed as a conjecture	proof was given	was known before	proposed as a conjecture	proof was given
Number of students	37	6	41	39	6	43	24	20	42
Property	B_4			B_5			B_6		
Status of the property	was known before	proposed as a conjecture	proof was given	was known before	proposed as a conjecture	proof was given	was known before	proposed as a conjecture	proof was given
Number of students	39	2	40	5	38	43	32	10	34
Property	B_7			B_8			B_9		
Status of the property	was known before	proposed as a conjecture	proof was given	was known before	proposed as a conjecture	proof was given	was known before	proposed as a conjecture	proof was given
Number of students	2	7	32	3	29	30	2	-	30
Property	B_{10}			B_{11}			B_{12}		
Status of the property	was known before	proposed as a conjecture	proof was given	was known before	proposed as a conjecture	proof was given	was known before	proposed as a conjecture	proof was given
Number of students	3	-	34	-	28	19	2	4	19
Property	B_{13}			B_{14}					
Status of the property	was known before	proposed as a conjecture	proof was given	was known before	proposed as a conjecture	proof was given			
Number of students	4	2	18	1	29	15			

Table 2.

Summary data collection of students' progress in increasing of their $AFKS_{bisector}$, $AFKS_{height,}$ and $AFKS_{median}$ in the first stage of the research is given in Table 3.

We would like to remark that didactical materials contained 14 properties for bisector, 16 ones for height and 12 properties for median correspondingly.

Properties	B	H	M	B, H, M altogether
Number of students, who proposed at least 12 conjectures	2	5	1	1
Number of students, who proposed at least 11 conjectures	3	6	1	1
Number of students, who proposed at least 10 conjectures	5	8	2	2
Number of students, who proposed at least 9 conjectures	7	9	7	6
Number of students, who proposed at least 8 conjectures	13	14	14	9
Number of students, who proposed at least 7 conjectures	14	16	17	13
Number of students, who proposed at least 6 conjectures	17	20	21	17
Number of students, who proposed at least 5 conjectures	29	30	31	28
Number of students, who proposed at least 4 conjectures	35	39	38	35
Number of students, who proposed at least 3 conjectures	38	41	41	36
Number of students, who proposed at least 2 conjectures	41	42	43	40

Table 3.

On the second stage we proposed the following task for students' work on their own:

"Find out properties of something using your previous findings and exercises for bisector, height and median".

As we mentioned above such task was aimed on developing and increasing of AFKS properly, using $AFKS_{bisector,}$, $AFKS_{height,}$ and $AFKS_{median}$ simultaneously.

Shortly, we would like to attract attention for students' strategy to produce conjectures, which was prepared in our teaching-learning environment and used by students on their own with full understanding and not by chance.

On the base of B_2 (see Appendix) 21 students proposed to construct different geometrical objects in a triangle. Thus, in that way 15 students came to the concept of symmedian of a triangle. After having a geometrical object, where content of their AFKS could apply to, 14 students produced a conjecture of existing a point of intersection for three symmedians of a triangle. As a result Lemoine point of a triangle was revealed. Property B_1 significantly contributed for students' inquiry work in that direction. At the same time property B_4 led to the students' conjecturing work (5 students did it successfully) for finding the following property:

"Symmedians of a triangle divide the opposite sides of it into the parts, which are proportional to the squares of the corresponding adjoining sides of this triangle".

It is important to note that other properties of Lemoine point were conjectured and proved by students in the similar way in our teaching-learning environment.

Final remarks

At the end we would like to emphasize peculiarities of students' conjecturing and proving activities in our research. At first, they do know direction of their work, i.e. a mathematical object for which they have been asked to produce conjectures, but the ways for achieving this aim are not indicated for students. At the same time initial information and further ideas can be taken from that properties, which were pointed out by students right away as known for them before. At last, directions of their further work are not indicated for students, but most of them are able to imagine how possible properties would look like.

Our results, although local, support hypothesis that most of the students can be successfully involved in the conjecturing and proving activities in different levels, if learning has active and constructive nature. In this work we have shown effectiveness of a constructivist approach to organise students' thinking leading to production of conjectures. We found out that a specific learning-teaching environment can significantly contribute to students' progress in learning geometry.

References

Anthony, G. (1996). Active learning in a constructivist framework. *Educational Studies in Mathematics*, 31, 349-369.

Arzarello, F., Gallino, G., Micheletti, C., Olivero, F., Paola, D. & Robutti, O. (1998). Dragging in Cabri and modalities of transition from conjectures to proofs in geometry. In: A.Olivier & K.Newstead (Eds) *Proceedings of PME22*, Stellenbosch, South Africa, Vol.2, 32-39.

Brown, S. I. and Walter, M. I. (1990). *Problem Posing*. Lawrence Erlbaum: Hillsdale NJ.

Davis, R. (1991). Constructivist views on the teaching and learning of mathematics. *Journal for Research in Mathematics Education*, Monograph 4, National Council of Teachers of Mathematics.

de Villiers, M. (1996). *Some adventures in Euclidean geometry.* Durban: University of Durban-Westville.

Edwards, L. (1997). Exploring the territory before proof: Students' generalizations in a computer microworld for transformation geometry. *International Journal of Computers for Mathematical Learning*, 2, 187-215.

Furinghetti, F. and Paola, D. (2003). To produce conjectures and to prove them within a dynamic geometry environment: a case study. In N.A.Pateman, B.J.Dougherty and J.T.Zilliox (Eds) *Proceedings of the 2003 Joint Meeting of PME and PMENA,* USA, Vol.2, 397-404.

Hanna, G. (2000). Proof, explanation and exploration: An overview. *Educational Studies in Mathematics*, 44 (1), 5-23.

Harel, I. and Papert, S. (1990). Software Design as a Learning Environment. *Interactive Learning Environments*, 1(1), 1-33.

Hiebert, J. (1992). Reflection and communication: Cognitive considerations in school mathematics reform. *International Journal of Educational Research*, *17*, 439-456.

Italian Ministry of Education. (2004). *New mathematical standards for the school from 5 through 18 years.* MIUR.

Mariotti, M. A. (2000). Introduction to proof: the mediation of a dynamic software environment. *Educational Studies in Mathematics*, 44 (1), 25-53.

National Council of Teachers of Mathematics. (2000). *Principle and standards for school mathematics.* Reston, VA: NCTM.

Ukrainian Ministry of Education and Science. (2003). *Programmes for schools. Mathematics*. Kiev, UMES.

Wang, M. C., Haertel, G. D. and Walberg, H. J. (1993). Toward a knowledge base for school learning. *Review of Educational Research, 63*, 249-294.

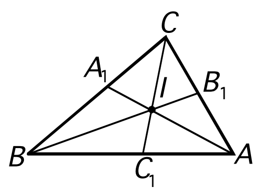
Yevdokimov, O. (2003). Intuitive proofs as a tool for development of student's creative abilities while solving and proving, In N.A.Pateman, B.J.Dougherty and J.T.Zilliox (Eds) *Proceedings of the 2003 Joint Meeting of PME and PMENA,* USA, Vol.1, 264.

Yevdokimov, O. (2004). Using materials from the history of mathematics in discovery-based learning, *Proceedings of the History and Pedagogy of Mathematics ICME-10 Satellite Meeting*, Sweden, 457-465.

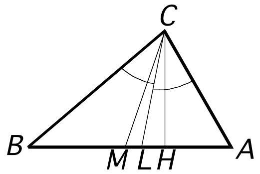
Appendix

Bisector's properties

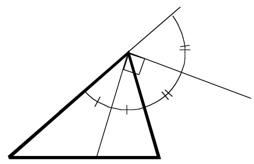
B$_1$. All bisectors of a triangle intersect in one and the same point, it is a center of the circumference, which is inscribed in that triangle.



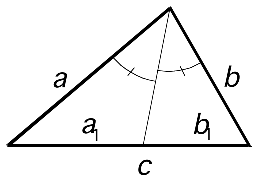
B$_2$. A bisector is between height and median from the same vertex of a triangle. In isosceles triangle bisector, height and median coincide.



B$_3$. Bisectors of interior and exterior angles of the same vertex of a triangle are perpendicular.



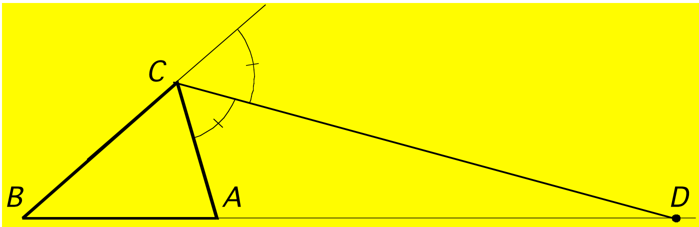
B$_4$. The bisectors of a triangle divide the opposite sides of it into the parts, which are proportional to the corresponding adjoining sides of this triangle. $\frac{a_1}{b_1} = \frac{a}{b}$.



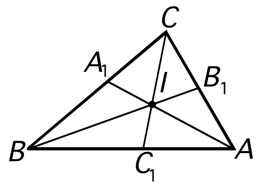
B$_5$. The bisector of the triangle with sides a, b, c divides the opposite side c on the segments $a_1 = \frac{ac}{a+b}$, $b_1 = \frac{bc}{a+b}$.

B$_6$. If a segment, which connects a vertex of a triangle with a point on the opposite side of that triangle, divides the opposite side into the parts, which are proportional to the corresponding adjoining sides of this triangle, then it is a bisector.

B$_7$. If *BD* is a bisector of exterior angle *C,* then $\frac{BD}{AD} = \frac{BC}{AC}$.



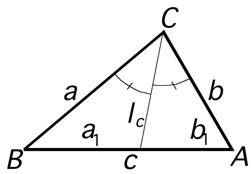
B₈. If bisectors of a triangle intersect in the point I, then it divides bisector CC_1 in the following relation $\frac{CI}{IC_1} = \frac{a+b}{c}$.



B₉. Length of bisector

1) $l_c = \dfrac{2ab\cos\frac{C}{2}}{a+b}$;

2) $l_c^2 = ab - a_1 b_1$.

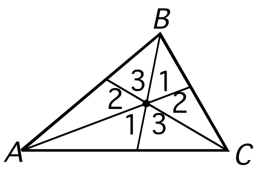


B₁₀. Angles between bisectors:

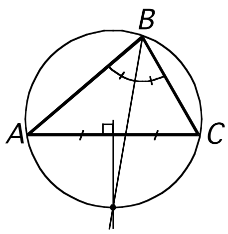
$\angle 1 = \frac{A+B}{2}$; $\angle 2 = \frac{A+C}{2}$; $\angle 3 = \frac{B+C}{2}$;

$\angle 1 + \angle 2 = 90° + A/2$;

$\angle 1 + \angle 3 = 90° + B/2$; $\angle 2 + \angle 3 = 90° + C/2$.

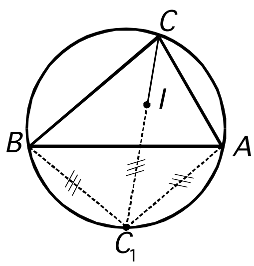


B₁₁. A bisector of a triangle and a mid-perpendicular to the opposite side of a triangle intersect in a point, which belongs to the circle described around this triangle

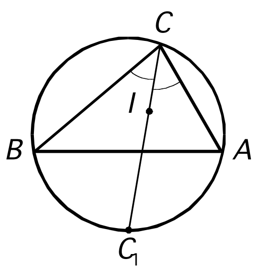


B₁₂. If point I is a centre of inscribed circumference into triangle ABC and point C_1 belongs to described circumference around this triangle and line CI simultaneously, then

$$C_1A = C_1B = C_1I = 2R\sin\frac{C}{2}.$$



B₁₃ If point I is a centre of inscribed circumference into triangle ABC and point C_1 belongs to described circumference around this triangle and line CI simultaneously, then $CI \cdot IC_1 = 2Rr$.



B₁₄. A bisector divides an angle between radius of described circumference around that triangle and its height from the same vertex of the triangle on two equal parts.