PROOF IN SWEDISH UPPER SECONDARY SCHOOL MATHEMATICS TEXTBOOKS - THE ISSUE OF TRANSPARENCY

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Abstract

We have investigated proof in two sets of commonly used Swedish upper secondary school mathematics textbooks. The frequency of proof items is low in each mathematical topic, even in the domain of geometry where pupils traditionally have learned proof. We explore the proof items with respect to different aspects of proof and discuss how they relate to students' access to proof. We show with some examples how proof often exists invisible in the textbooks and discuss the difficulty of giving a correct definition of proof at upper secondary school level.

1. Introduction

The role of proof in the Swedish schools has diminished during the last twenty years, a development, which is similar to that of many other countries (Hanna, 1995; Waring, 2001).

The Swedish curriculum for upper secondary school does not clearly state the aims of introducing the students to proofs and proving activities. Only the main goals are stated. "The school in its teaching of mathematics should aim to ensure that pupils develop their ability to follow and reason mathematically, as well as present their thought orally and in writing." (Skolverket, 2002, p. 60) Local schools and teachers have the possibility of applying these goals in their own way. However, one of the Criteria for 'Pass' (lowest mark of a three-level grading scale: Pass, Pass with distinction, Pass with special distinction) for any of the five courses A-E, into which upper secondary school mathematics is divided, is that "pupils differentiate between guesses and assumptions from given facts, as well as deductions and proof" (pp. 60-66). Furthermore, one of the 'Criteria for Pass with special distinction' is that "pupils participate in mathematical discussions and provide mathematical proof, both orally and in writing" (pp. 60-66)

Most of the university entrants, who responded to the survey conducted by Nordström (2002), stated that they had had little experience about proofs in upper secondary school, especially in form of own practice. The aim of this textbook study is to complement the survey in order to create a more varied picture of students' school background concerning their experiences about proof.

Researchers have paid relatively little attention to the role of the textbooks in the teaching and learning of proof (Hanna and de Bruyn, 1999). “In particular, there has been almost no examination of the actual occurrence in mathematics textbooks of proofs, discussions of proof, and exercises requiring the construction of proofs.” (p. 180) Many studies show that textbooks and teacher guides influence the teaching practices and the choices of the contents being treated in the lessons. This is particularly true for mathematics (Englund, 1999).

The first impression when skimming through some Swedish upper secondary school textbooks is that the frequency of proofs and proving tasks is minimal. However, we wanted to systematically explore the occurrences of proofs and proving tasks in different mathematical domains. Our aim was also to analyse the nature and level of the proofs included in the textbooks and the manner in which the proofs were presented to the reader. The main question is how the textbooks enhance the students’ access to proof and in what way the textbooks help students to develop their understanding of different aspects of proof.

2. Theoretical frame

Our approach to the issue is socio-cultural. According to the socio-cultural theory, learning is an aspect of interrelated historical, cultural, institutional and communicative process (Renshaw, 2002)

2.1. Proof as an artefact

The notion of transparency of artefacts as Lave and Wenger (1991) define it is central for the study. We consider proof, in a very similar way as Adler (1999) considers talk when studying multilingual mathematics classrooms, as a *resource for mathematical learning*. Then proof needs to be both seen (be visible) and to be used and seen through (be invisible) to provide access to mathematical learning. We argue that Lave and Wenger’s concept of transparency captures this dual function of proof as a learning resource in mathematics.

“…the notion of transparency, taken very broadly, is a way of organising activities that makes their meaning visible,…” (Lave & Wenger, 1991, 102).

Access to artefacts in the community both through their use and through understanding their significance is crucial. We study how different aspects of proof are visible/invisible in the textbooks.

2.2. Conceptual frame

We use the conceptual frame created by Nordström (2004) in the analysis of the data. The aspects in the frame are *Inductive/ Deductive approaches, Conviction/Explanation*, *Formality, level of rigour and the language*, and *Aesthetics*. We explore if the different aspects of proof are visible in the textbooks and discuss how they relate to students’ access to proof. For an account of these aspects see Nordström (2004).

2.3. A review of relevant research

We found only two studies dealing explicitly with proof in textbooks. Hanna and de Bruyn (1999) investigate the frequency of proof items in year 12 (aged17-18) advanced- level mathematics in Ontario. They found that only in the topic of geometry, textbooks do a reasonable job of providing opportunities to learn proof and a range of proof types was presented. No visual proofs, heuristic arguments or the presentation of correctly or partially completed proofs were used for critique by the student.

Grenier (2000) studies the status and role of proof in 11-16 years pupils' curricula and schoolbooks in France. She finds a discrepancy between the curriculum goals and the contents in the textbooks. She remarks that even if the curriculum gives possibilities to employ new ways of treating proof, the proof practice seems to live primarily in the field of geometry and under two forms: 1) Simple deductive reasoning (with a small number of steps), based on the use of statements or theorems from the theoretical part of the course. 2) Writing proofs, with more attention given to language and syntax rather than logic and semantics.

Of course, there are a lot of studies where proof is considered in textbooks but not as a main issue (e.g. Bremler 2003). Bremler studies derivative in the textbooks during 1967-2002. His thesis is of interest for us because it concerns the Swedish upper secondary school textbooks. He briefly describes some changes in the treatment of proof and states that after 1994 (when a new curriculum was implemented) there is no proof of the general rule for derivative of a polynomial in any of the textbooks. The number of textbooks that neither show nor mention the proof has doubled since 1994. However, Bremler (2003) points out that some of the textbooks have preserved the proof but as an exercise with hints.

3. Methodology

The upper secondary school mathematics in Sweden is divided into five courses from A to E. We chose two commonly used set of textbooks, covering the courses from A to E, *Liber Pyramid* and *Matematik 3000*. The topics in both sets are: Algebra, Geometry, Statistics and probability, Functions and calculus, Exponents and logarithms, Trigonometry and Complex numbers. The textbooks are divided in chapters and sections within chapters. The sections contain explanatory text and examples, followed by exercises (with hints and solutions). Most of the exercises are divided into three levels, which correspond to the different marks in a three-level grading scale: Pass, Pass with distinction, Pass with special distinction.

When designing the rational of the study one of our aims was to examine the frequency of proof items in the textbooks as Hanna and de Bruyn (1999) did in their study on Ontarian textbooks. We examined the textbooks for the frequency of occurrence of various kinds of proofs and of items related to constructing a proof. Very soon though, we realised that the occurrence of proof or discussion on proof was very low compared to the Ontarian textbooks. Furthermore, proofs were seldom made explicit in the explanatory sections. It was not reasonable to use the quantitative method which Hanna and de Bruyn (1999) developed in their study. Therefore, we decided to work mainly with qualitative content analysis. We counted the percentage of the proving tasks, studied the textbooks, chapter by chapter and picked the items that have some significance for learning of proof. Afterwards, we related these items to the research questions, earlier research on proof in the textbooks and to the conceptual frame created by Nordström (2004). During the exploration we created some posteriori subcategories and checked the textbooks again with respect to them.

4. Results

We start with reporting to what extend proving tasks occur in different mathematical domains and go on with the different aspects of proof. All these results have some significance to students' access to proof and the notion of transparency, to which we particularly denote the last section of the chapter.

4.1. Frequency of different kinds of proving tasks

The space given to proving tasks is minimal compared to practical applications and routine exercises (about 2%). However, there are some special mathematical domains where proving tasks are more common: in geometry, in the context of verifications of solutions of differential equations and verification of formulas of trigonometric functions.

Example: *Show that* $2\sqrt{x}$ *is a solution for differential equation* $2xy' - y = 0$.

The pattern is the same as in the Ontarian textbooks, even if the percentages of the proof occurrences in Ontarian textbooks were substantially higher in geometry. Of all tasks in the geometry chapters in the Swedish textbooks, about 10% are proving tasks compared to the two Ontarian textbooks where 56% respectively 40% of the geometry tasks were proving tasks (Hanna and de Bruyn, 1999). Also in the French textbooks proof was concentrated in geometry (Grenier, 2000). In the two first courses (A-B), geometry is the only area where some proofs are explicitly given in the explanatory sections.

There are a few tasks dealing with counterexample. There are no indirect proofs presented in the ordinary course in either *Matematik 3000* or *Liber Pyramid*. In the Ontarian textbooks the concept is introduced in the geometry chapter and there are some exercises on it (Hanna and de Bruyn, 1999). Tasks encouraging pupils to make conjectures and own investigations are unusual in the Swedish textbooks. This was something Grenier (2000) also found out in the French textbooks.

There are some examples of potentially productive tasks for enhancing students' understanding of proof, but we argue that they are occasional and not systematically presented. Further, the range of the types of proofs given in the explanatory sections or as exercises are too narrow to be representative of mathematical practice. This is something that Hanna and de Bruyn (1999) also noticed in their study. The textbooks do not offer practice enough for the students to learn to construct different kinds of proofs.

4.2. How the different aspects of proof are visible in the textbooks

4.2.1. Inductive/ Deductive approaches

Students have often difficulties in dealing with algebraic symbols. That is why it is an important task for the teaching of mathematics to accustom the students to algebraic symbols and help them to realise the generality and the power of using such symbols. We found that a clear distinction between examples and general proofs is not always made visible. Below follows four different modes we found in *Liber Pyramid* and *Matematik 3000* of introducing some rules and formulas:

(1) The rule is given through some specific examples.

This mode is used when introducing the rule for multiplication of negative integers in *Liber Pyramid* for course A. In *Matematik 3000* an explanation of the role is given by a generic example. In *Liber Pyramid* this kind of explanation is given in a subsequent course, i.e., the C-course.

(2) The general rule is given with algebraic symbols followed by a special case as an example.

(3) The rule is introduced by generic examples.

This mode is quite usual in both of the textbook series. After the examples, the authors write: 'Generally (holds)…' , 'It is possible to show that generally…', 'One can strictly show…', 'In the same way one can prove'… , 'We can obviously formulate the following rule…', 'Without a proof we accept the following rules…'. Sometimes these comments are lacking and no clear distinction is made between the special and the general.

Example (*Liber Pyramid*):

$7^2 \cdot 7^4 = (7 \cdot 7) \cdot (7 \cdot 7 \cdot 7 \cdot 7) = 7^6$

So we get $7^2 \cdot 7^4 = 7^{2+4} = 7^6$

In the same manner we get that $3^3 \cdot 3^5 = 3^{3+5} = 3^8$

Generally holds $a^m \cdot a^n = a^{m+n}$

Also, in *Matematik 3000* exponent laws are introduced in the same way. In *Liber Pyramid*, a general proof is demanded as a middle-level exercise in the same section. However, the difference between the generic example and the proof is not visible. The proving task starts by 'Explain that...'. We think that the generic example can already be seen as an explanation. Then it would be more honest to say '*Prove* that it holds for all positive integers'. In the solutions a complete algebraic proof is given.

(4) The general rule with a deductive proof is given.

There are deductive proofs in the textbooks but they are not always called proofs, especially in the textbooks for the two first courses. Later, they are sometimes called proofs, sometimes derivations. Very often a specific or a generic example is introduced in parallel to the proof but often without an explanation of what the general and what the specific is. There is one example when the authors first test some properties with specific examples and then investigate if 'it is possible to show the general case'. There are some tasks encouraging pupils to make conjectures and own investigations but they seldom lead to a construction of a deductive proof.

4.2.2. Aspects of Conviction/Explanation

The proofs in the textbooks are obviously not given as obligatory rituals but in order to explain why the statement is true. In *Liber Pyramid* the proofs in the first textbooks are even presented as synonyms for explanations. However, some of the proving tasks start with 'Make believable that… (Troliggör att…)'.

4.2.3. Aspects of Formality, level of rigour and language

In contrast to the French textbooks (Grenier, 2000), the aspects of writing proofs are not dealt with at all. The style in proofs in the explanatory sections and in the solutions and hints is everyday language rather than formal mathematical language. Words like *proof, definition, assumption* are generally avoided in the textbooks, especially in the first two courses (A-B). This aspect can have some significance to the students' access and transparency because there are very few examples on how to write proofs in the explanatory sections.

When trying to avoid the word proof a confusion of different notions is easily created. In *Liber Pyramid*, proof is first defined as "logical reasoning without gaps". The word *explanation* is used later, instead of the word proof in the beginning of the book. Later the authors tell that the Greeks *discovered* that $\sqrt{2}$ was an irrational number, instead of the Greeks proved or showed that $\sqrt{2}$ was an irrational number. In *Liber Pyramid* for course A the derivations of the formulas for areas of polygons are called *justification (motivering)*. However, the proof of the area for trapezoids has not even a heading *justification* and the proof is hidden in the text.

The textbooks do not enhance the students' abilities to *differentiate between guesses and assumptions from given facts, as well as deductions and proof*, which is one of the curriculum goals for all the students. The words like assumption, guess/conjecture, given facts or deduction are not even mentioned. There are no exercises that shed light on the meaning of definitions. For example, in the geometry chapter in *Liber Pyramid*, the word definition is not used at all even if a definition of the concept 'definition' is given in the summary of the same chapter.

4.2.4. Aesthetics

We found two sentences referring to aesthetic aspects of proofs and theorems, both of them in *Liber Pyramid*: 'This beautiful and powerful formula can be used to…' (de Moivre's formula). 'Pretty and useful!' (the formula for a geometric sum). Aesthetics is an aspect that mathematicians often refer to when talking about proof (Nordström, 2004). So it would be natural to make also the aesthetic aspects of theorems and proofs visible in the textbooks.

4.3. Transparency and the students' access

Differentiating the tasks in the textbooks is made in a way, which indicates that the textbook authors consider that proofs are accessible only for already motivated and high achieving pupils. The proving tasks are most often placed in the second or third level of difficulty. This is in harmony with the criteria of judgement, which the Swedish curriculum states. However, an exception is the chapters of trigonometric formulas and differential equations, where examples on how to verify equalities are given in the explanatory section followed by exercises at all levels of difficulty.

None of the textbooks included in this study pursued to make proofs visible in a manner that focus on the logic and the structure of the proofs. However, there are some slight differences between the two sets. *Matematik 3000* offers some examples on how to prove some geometrical statements and contains more proving tasks in geometry than *Liber Pyramid*. There is also a rational order in *Matematik 3000*, in which the examples and the tasks follow each other and the structure gives the students a possibility of building deductive chains and start from clear theorems or axioms. That is not always the case in *Liber Pyramid* where you for example meet a middle level proving task that demands the knowledge of the similarity of triangles that is given before, but outside the ordinary course. Even if *Matematik 3000* offers some structure in presentation of proofs there are examples on how proof is made invisible as well. For instance the word 'proof' is omitted when presenting theorems and proofs.

Unlike *Matematik* 3000, where we could not find any discussions or explanations on proof or theorem, the authors of *Liber Pyramid* make some attempts to explain these notions. In the connection to the theorem *The sum of the angles in a triangle is 180°* the following explanations are given in *Liber Pyramid* (our translation):

'A statement that is true is often called a *theorem* in mathematics, e.g. Pythagorean theorem.'

'An explanation for why the theorem is true is called a *proof.*'

The following definitions are from the summary in the end of the geometry chapter in *Liber Pyramid* (our translation).

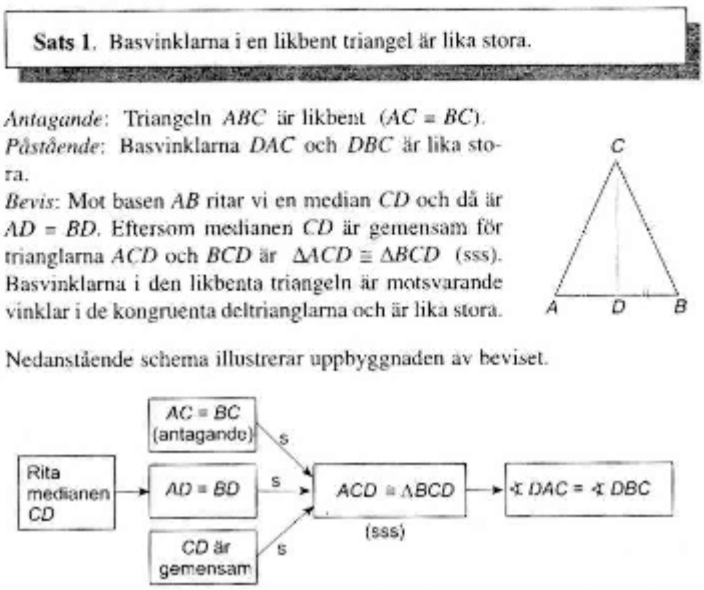
Theorem	A statement which has been proved. For example: The Pythagorean theorem.
Proof	A logical reasoning without gaps.
Axiom	A statement that is accepted without a proof. For example: The Parallel axiom in Euclidian geometry.
Definition	Determines (explains) what something is. For example: The definition of a right triangle.

We notice that there are different definitions for proof and theorem in the same textbook.

5. Discussion

An interesting theoretical question is to what extent and by what means it is possible to make the different aspects of proof visible at the upper-secondary level. The authors of *Liber Pyramid,* for example, attempt to explain the fundamental notions *statement, theorem, proof* and *truth* but as we have shown, it is rather confusing. This is not surprising because there is not a consensus on these notions among all mathematicians. The definitions are different in *classical* mathematics and *constructive* mathematics (e.g. Richman, 1997). In classical mathematics a *statement* is something which is true or false. A *proof* of a statement is a logical reasoning which makes it evident that the statement is true. The notion of *truth* is left undefined. In constructive mathematics, a *statement* is defined if one knows what a *proof* of the statement consists of. A statement is *true* if there is a proof of the statement. Hence truth is defined as provability and this is in fact implicitly also done in *Liber Pyramid*, since they have the following two definitions of *theorem*: 1) a true statement, 2) a statement which is proved. Students consider the act of proving theorems as more cumbersome than the act of computing. In the textbooks there are no rules for proving and no discussion of how proofs are created and also very few examples. In constructive mathematics there is no real difference between the activities of proving theorems and of computing (Martin-Löf, 1985). It is possible that knowledge of this fact could contribute to the way these notions are treated in school. This could be a subject for further investigations.

Even if it is difficult to give a correct definition of proof there are several ways of enhancing students' understanding of proof. The following example from a Finnish textbook shows how, for example, the structure in geometrical proofs can be made more visible. In the textbook above the figure, there is first a discussion on how to find out from the formulation of the theorem (the base angles in an isosceles triangle are equal) what is the assumption and what is the statement, which is not necessarily easy for the students to decide. The proof then begins with the *assumption* (*Antagande*): 'The triangle ABC is isosceles.' It follows by the *statement* (*Påstående*): 'The base angles *DAC* and *DBC* are equal.' After the proof the logical structure of the proof is illustrated with a figure. Such figures are employed also in the exercise sections and in all the following geometrical proofs given in the explanatory section. The figures illuminate the process of proving by showing its logical structure and how the necessary arguments needed for the conclusion are obtained from the assumptions, definitions, constructions, axioms or theorems.



Figur 1 Geometri, Gymnasiematematik, lång kurs, p. 88

6. Conclusion

The results of our study can be understood when considering them against the historical background. In the Swedish national curriculum 1994, the word proof was not mentioned. The style of making proof invisible in the textbooks can be a reaction against the earlier, very formal way of presenting theorems and proofs. It is not an easy task to decide how to treat proof at this level. However, the recent curriculum states some goals concerning proof and it is important to discuss how the textbooks meet the new requirements.

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