A GENETIC APPROACH TO PROOF

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Abstract: The paper proposes the metaphor of a 'Theoretical Physicist' for explaining the thinking of students in proof situations and for designing a genetic approach to proof. In regard to students' cognitions, a formative and an established phase in the (personal) development of a theory are distinguished. In the genetic approach to proof which is sketched in this paper, geometry is, during an introductory phase, treated as an empirical theory.

The Metaphor of a "Theoretical Physicist"

Quite a few students of the secondary and even tertiary levels tend to confirm mathematical statements by appealing to measurements or special examples. They consider them as sufficient argument though they already have had experiences with mathematical proofs and, according to their teachers, should know that the general validity of a mathematical statement cannot be established in this way (see for example Healy & Hoyles, to appear; Coe & Ruthven, 1994). Some papers report on students asking for additional testing of a statement by means of an example after its proof has been treated in the classroom and and in spite of the fact that the students claim to have understood it. (Fishbein, 1982, 16). Some students believe that a proof is valid only for the special diagram which is attached to it (Hefendehl-Hebeker, 1995).

In the present paper I will search for explanations of these phenomena and try to show that this type of thinking occurs with a certain necessity and is a consequence of a meaningful behaviour. In a second step some proposals will be made for dealing with these ideas of the students in a constructive way and for leading them step by step to mathematical argumentation proper. I call this a 'genetic approach to proof'.

Frequently, difficulties of students with proof are considered as indicating a lack of logical insight (see f.e. Stein, 1990). Therefore, some textbooks introduce the topic 'proof' by exposing logical ideas as, for example, the difference between a statement and its converse and the distinction between a statement which is true for some cases and a generally valid one. In contrast to this, the present paper departs from the hypothesis that 'mathematical proof' is above all a *problem of epistemological meaning* (Hanna & Jahnke 1993, 433f). The role and function of mathematical proof will not be explained to the students by referring exclusively to mathematics and to purely mathematical examples. Rather, the contribution of mathematical proof to human cognition in general and to human understanding of the surrounding world has to be exposed. Put in another way, I will consider mathematical proof not through the eyes of a pure mathematician but from the point of view of a *theoretical physicist*. The term 'theoretical physicist' is meant metaphorically and designates a person who develops and evaluates mathematical deductions in order to better understand the world in which he lives.

Using this metaphor we bring ourselves in a position beyond the established division of labour which separates mathematics from other disciplines. Especially, we get a fruitful alienation of our view on the relation of mathematical proof and special cases. While in pure mathematics we believe in the general validity of a theorem established by a mathematical proof, the situation in physics is different. Physicists will not accept a conclusion of a theory derived by a mathematical argument without experimental verification. If a conclusion is new and important, a physicist will develop an experimental test of it. In this sense, the metaphor of a 'theoretical physicist' might help to analyse the thinking of a pupil who prefers empirical arguments.

Proof and Measurement in Mathematics Teaching

Students are frequently asked to measure the angles of a triangle, for example. After they find that their sum is always nearly equal to 180 degrees, they are told that measurement can establish this fact only for individual cases and that they will have to prove it if they really want to be sure that it is true for all triangles. This explanation may have been obvious at the time of Plato and Euclid, but it must seem unsatisfactory at a time when experiment and measurement are considered the foundation of scientific methodology. To avoid contradiction, the students are, of course, told that the triangles they draw are fundamentally different in nature from the triangles that play a role in geometrical theorems - the latter are ideal or theoretical entities, whereas the former are empirical. As valid as this distinction may be, teachers themselves still assume that one can be sure about the sum of the angles of empirical triangles, without any further reflection, after the respective theorem for ideal triangles has been proved, and they cannot help conveying this conviction to their students.

In physics lessons, however, the message is exactly the other way round. One would never seriously entertain the idea that a natural law might be established by a theoretical proof, and one would insist that all such laws are founded upon experiment and measurement. (see Hanna & Jahnke, 1996, 892 f).

Therefore, in teaching beginners an intellectually honest way is to take side by the physicist and to say that the angle sum theorem is true because of empirical measurements. Only at a later stage, one should expose the idea of a purely mathematical theory separated from reality.

In the next section we will explain the consequences of this position and the meaning of mathematical proof in empirical theories.

Proof and Measurement in Physics

Theories of physics are systems of statements/theorems whose quantitative consequences are expressed in natural laws. Natural laws are mathematical equations between measurable magnitudes, for example the law of gravity or the dependence of the product of pressure and volume on (absolute) temperature in ideal gases. The ensemble of natural laws contained in a theory can be derived from a few fundamental assumptions by *mathematical proofs*. Within a certain domain of tolerance the validity of the natural laws is corroborated or refuted by experiments/measurements.

An isolated natural law which is not linked to other natural laws can only be corroborated by experiments which are directly related to it. In contrast to this, a natural law within a theoretical network is corroborated not only by direct measurements but by all the measurements of all the natural laws belonging to the network. Thus, a mathematical proof deriving a statement (natural law) of the theory does not provide absolute certainty to this statement, but it will considerably *enhance* its certainty since the corroboration does not only come from an isolated set of direct measurements but from all measurements related to the theory. Such a theoretical network of statements and measurements connected by mathematical proofs is the safest form of knowledge at our disposal, though this does not change its, in principle, preliminary character.

In the development of science the firmness of a theory becomes especially visible in such moments when a mathematical deduction allows a not obvious prediction which afterwards is corroborated by measurements. The ability of a theory to make such predictions is the most important criterion of its firmness and fruitfulness.

Applied to our example of the angle sum theorem this means that its proof will not give absolute certainty to the theorem, but it will enhance its certainty since it connects the theorem with other geometrical theorems which can also be corroborated by measurements. Thus, the theorem is not only tested by measuring the sum of the angles of a triangle, but also by measuring corresponding angles, alternate angles or sums of angles in 4-, 5-, 6-gons. In a long school career the students come to know so many statements which are connected with the angle sum theorem and empirically testable that they understand why Euclidean geometry is the eldest and empirically best corroborated theory we have and why, for a long time, mathematicians and philosophers attributed absolute certainty to it.

To fully understand the problem of justification we have to discuss a further idea which is especially stressed by *holistic* philosophies of science. They point to the fact, that the methods for measuring the magnitudes in a natural law suppose, as a rule, the validity of the theory containing this law. For example, it is not possible to measure the magnitudes force (F) and mass (m) appearing in Newton's law $F = m \cdot a$ without supposing in some way that this law is already valid (see Jahnke, 1978). Philosophers of science say that the observational language is 'theory loaded'. P. Duhem (1908/1978) was among the first who pointed to the importance of this fact.

As a consequence holistic philosophies of science from P. Duhem to J. D. Sneed (1971) claim that *theories as a whole* have to be corroborated or refuted. If we accept this claim, and I think there is no reasonable alternative to this, then we have to draw the conclusion that a theory has to be *jugded* whether it is *successful*. In the process of assessment many criteria and points of view play a role which, in part, are beyond the limits of explicit reflection. All in all this assessment is a matter of judgement and, thus, a *pragmatic decision*. The ultimate reason for accepting a law is not of a logical, but of a *pragmatic nature*.

Assessing a theory as being valid or acceptable implies a statement about its *future*. Scientists express their *expectation* that it will be possible to derive further phenomena and applications from the theory at hand which will be corroborated by experiment. However, scientists are always conscious of the possibility that new phenomena might be discovered which falsify the theory in its present form and require to modify some laws or even to introduce new parameters. In principle, scientific theories are open to revision.

I would like to call this fundamental fact the *circle of justification*. In regard to the angle sum theorem we have to conclude that a logical proof alone cannot provide certainty to it. A logical proof reduces a theorem only to other theorems as for example to the theorem about alternate angles at intercepted parallels. The latter, however, has no higher degree of certainty than the former. The statement "In future, I will consider this theorem as valid" requires an assessment of the whole situation independent and different of the mathematical proof. This assessment takes into account results of measurements, other theorems, considerations of plausibility. Such phases of assessment are not part of the teaching of 7^{th} graders to whom the angle sum theorem is conveyed. The underlying concept of proof which is determined by a purely logical view excludes this type of reflection.

A judgement about a theory as valid and successful, however, can only be made in an *advanced* and *late* period of its development. Consequently, we have to distinguish between two phases, that before and that after this judgement. We call the former the *formative* phase of a theory, the latter its *established* phase. In the formative phase a theory is considered as one among several possibilities of explaining a certain area of phenomena. It is not clear which the adequate basic concepts are and the theory is taken as hypothetical. In the established phase the theory has been judged as being valid and successful and is taken as the only legitimate explanation of the phenomena to which it is related. Scientists agree upon definitions of the basic concepts, the explanations of the theory are considered as safe, the theory has been transformed into a "system of theorems which can be derived from a few axioms".

The Different Functions of Proof in the Formative and Established Phases of a Theory

Mathematical proofs have different functions in the formative and established phases. When Newton published his *Philosophiae naturalis principia mathematica* Kepler's laws of planetary motion were well-tested empirical statements whereas Newton's law of gravity was an uncertain and to a high degree contestable hypothesis. Newton's proof, thus, couldn't have the function to establish the truth of Kepler's laws. Rather, it was the other way round. The fact that Kepler's laws could be derived from the law of gravity was the decisive argument in favour of the latter. To use a term of I. Lakatos Newton's proof didn't effect a flow of truth from the assumptions to the conclusion, but, vice versa, from the conclusion (Kepler's ellipses) to the assumption (the law of gravity). In the last regard, Newton's proof was a 'proof' of the law of gravity.

However, the situation was even more complicated and this sharpens our stance. At Newton's time astronomers were well aware that Kepler's laws did not exactly describe the movements of the planets. Therefore, astronomer Cassini proposed certain ovals (4^{th} degree curves) instead of ellipses as paths of the planets. Some astronomers followed him. In this situation Newton's proof was a strong argument in favour of Kepler's ellipses and, thus, there was also a flow of truth from the assumptions to the conclusion.

However, in order to arrive at this conclusion the scientists at that time had to *evaluate the situation as a whole*. It was the *fit,* or, to say it more emphatically, the *harmony* between Kepler's laws and the law of gravity which served as the decisive argument to accept both of them at the same time as the adequate and 'right' theory of planetary motion. In a second step the deviations of a planet from the elliptic path were explained as perturbations by the other planets, in order to get an empirically satisfactory theory.

Therefore, in the formative phase of a theory proofs do not effect an uniquely directed flow of truth since there are no accepted foundations. What a proof means in a concrete situation is subject to a judgement which has to take into account the whole theoretical and empirical context.

The Behaviour of Proof Novices

Proof novices are considered in this paper as students in their personal formative phase of a geometric theory.

Semi-formal interviews with 6 students of grade 7 which took place after a first introduction into geometrical proof showed some interesting results. 4 students were ready to consider the possibility that there are triangles with an angle sum different of 180° degrees. Rather, the low achievers denied such a possibility (see Balacheff (1991) for similar experiences).

Being asked to prove the angle sum theorem none of the interviewed students gave the Euclidean proof though this proof had been treated in class and was required in a written test a week ago. However, five students were able to reproduce the Euclidean proof after they had been shown the figure related to this proof. Instead of the Euclidean proof four students derived the angle sum theorem from the fact that an exterior angle of a triangle is equal to the sum of the non-adjacent interior angles.

These findings show: the students move tentatively in a network, but the network is still lacking structure. They are still in the formative phase and behave accordingly.

Another example from teaching is as follows. After a proof of the fact that the perpendicular bisectors of the sides of a triangle meet in one point a pupil remarked: “You can’t say that this is true for every triangle, there are still other triangles [pupil points to the drawing]”. The teacher answered: “We have only used properties valid in every triangle” and continued with another subject (Hefendehl-Hebeker 1995; see also Williams, 1979, and Balacheff, 1988, for the observation that, frequently, students think a proof valid only for the triangle shown in the drawing). Clearly, the pupil in this episode is in the formative phase of a theory. He is not sure whether there has not been used some hidden or not explicitly mentioned feature of a triangle by referring to the drawing. Also, there could be some parameters involved which are not known for the moment but which might influence the situation. Whether a drawing is special or general and whether one uses a general drawing in a special way is a matter of an *evaluation* which presupposes experience with proofs in Euclidean geometry, but is not part of the proof and not at all a matter of pure logic. Thus, the conclusion that the proof is valid for all triangles presupposes an evaluation of the whole situation which involves pragmatic aspects. On the other hand, the teacher is in the established phase of the theory and argues from his point of view. To him it is obvious that the parameters used in Euclidean geometry (length, angle) determine the situation completely, and he has a sense, built up by experience, which aspects of a figure are general and which are not. However, his inability to realise that he is arguing from a point of view completely different to that of his pupil causes a serious problem of teaching and of understanding the nature of proof in general.

The Genetic Approach

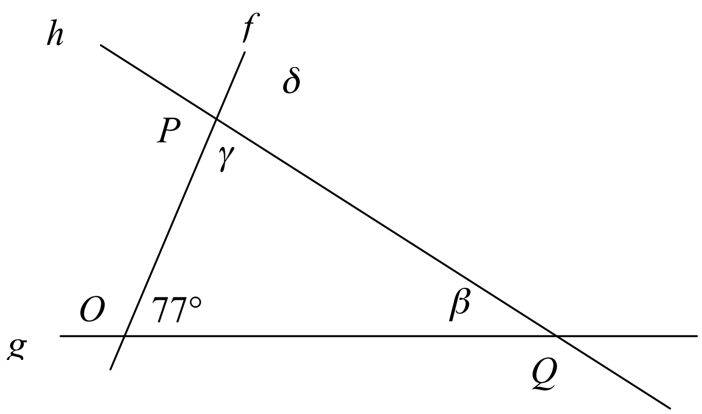
The point of departure for introducing pupils to mathematical argumentation is the question ‘why’. A regularity or pattern is observed and the question arises what makes things the way they are. Answering such questions can be considered like performing a *thought experiment*. The basis of argumentation and the chain of conclusions have to be developed simultaneously. Such activities can already occur at the elementary level (for examples see Wittmann & Müller, 1990)

After such a ‘culture of why questions’ has been developed over several years of mathematics teaching there will come a time when proof is explicitly taught and the term ‘proof’ explicitly mentioned to the students. In Germany, this is usually the case in grade 7 in the course of geometry teaching. One of the first proofs is that of the angle sum theorem for triangles.

Usually the theorem is proved in the classical Euclidean way and reduced to the alternate angle theorem. The latter is ‘proved’ by means of transformation geometry. Mathematically, this is not terribly honest since the axiom of parallelism is tacitly used and not mentioned to the students. Obviously, the intention of text book authors and teachers is to convey to the students the idea that proof leads to absolute certainty in contrast to measurements which are not precise and valid only for single cases.

In contrast to this, I would suggest that –at this stage!- geometry is presented as an *empirical theory.* Then the meaning of proof is not to replace measurement by something more certain, but to relate measurements and proofs in a theory. Proofs help to measure in an intelligent way (Winter, 1983; Jahnke, 1978).

To give an example I would propose to present the alternate angle theorem neither as a theorem proven nor as an evident truth but as an *uncertain empirical hypothesis*. Doing this we introduce the idea of modelling and the possibility of non-Euclidean geometries into teaching in a way adequate to pupils of the 7th grade. Consider the following work-sheet.



The straight line h rotates counter-clockwise around vertex P.
For some positions of h measure the angles β, γ and δ. Tabulate and write down your observations.

Students will observe: (1) Q travels more and more to the right. (2) γ increases, β decreases. (3) Losses of β are exactly compensated by gains of γ. (4) There is at least one position of h with $\delta = 77°$. (5) In this position h and g will not intersect. (6) To this position corresponds that Q has travelled to the infinite and that $\beta = 0°$.

It is plausible that the losses of β are exactly compensated by gains of γ also in the infinite (there shouldn't be a jump). But we cannot know this and we cannot test this directly. But we can prove: if exact compensation takes place then the angle sum of a triangle is exactly 180°.

To this idealised chain of observations and arguments the teacher could add the remarks that Euclid wasn't sure about exact compensation in the infinite and therefore added this statement as a hypothesis to geometry and that Gauss tried to test this hypothesis by measuring angles in very large triangles.

Having reached this starting point further theorems involving angles in triangles and general polygons might be proved and empirically tested. Theorems which are proved before any measurement has been done are like predictions in physics. In this way students come to study a 'small theory', and they get an idea of what a theory is, how measurements and proofs are related and why it is better to have a theory than not to have one.

In subsequent teaching further small empirical theories should be developed as it is done in the project "Arguments from physics in mathematical proofs" (see Hanna & Jahnke 2002) where for example axioms from static are the starting point for a theory.

In a final phase theories are constructed for their own sake. These theories are no longer considered to be empirical theories.

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