Visualising and Conjecturing Solutions for Heron's Problem

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Abstract

This paper analyzes with a practical teaching porpouse some aspects of using visual constructions during problem solving processes. In particular, we analyze some geometric constructions made to conjecture the solution to Heron's problem, and obtain two different categories of visual representations. The paper includes explicit contributions of some participants of group on proof at CERME IV, specially trying to make connections between research and teacher practice.

1 Introduction

Visualization has been a main topic in mathematics education research for the last decade, and many different theoretical perspectives and practical resources have been developed [16] [12] [17] [8]. Especially in the field of geometry many contributions have been made, emphasizing the connection between symbolic and visual representations of mathematical knowledge [10], [1].

Attending the specific moment of formulating a conjecture, many investigations have focused on the role of drawings within geometrical activity, and inquired into the intimate relationship between drawings and concepts [14].

The main theoretical ideas that we will consider here as a framework are the following:

1. Visualization is a concept connecting both symbolic and iconic representations, which are linked by the problem solver in a process of interaction. This view of visualization is characterised by a continuous change between these representations [10]. In the school context, the teachers often gives support to this change by acting as an external mediator between iconic and symbolic thinking.
2. Among the functions attributed to visualization, we focus here on its power as a builder of mathematical ideas, and we consider it necessary for heuristic reasoning [12], and thus of producing conjectures. The symbiosis between concepts and geometric figures stimulates new directions of thought, but there are logical or conceptual constraints which control the process [7].

3. Both, visualization and conjecturing have an important contextual component, and any observation is made within a particular context, so that its applications in other situations, surely vary [12], [5]. It is not our objective to produce some general classification of conjectures, able to be applied for every problem or any situation, but to analyze what in a very concrete setting occurs to build bridges between practice and research. For theoretical works supporting this link between teaching and research, see [2, 3, 13].

As we analyzed in this paper the kind of conjectures arsing from the discussion of Heron's problem, we pose it now, and comment briefly its solution:

Heron proofs in his *Catroptica* that the light rays cover a minimal distance supposed that incidence angle and reflection angle are equal [9]. A modern statement of the problem, more commonly used, is the following:

Let s be a line and A and B two points at the same side of s. For which point P in s is $AP + PB$ the shortest way joining A and B? [4]

The most usual geometrical construction used to solve the problem is the same proposed by Heron:

Let C be the point symmetric to A with respect to the line s, so that the segment $\overline{AC}$ is perpendicular to s. The line joining B and C intersects s in P, which is the point we are looking for. (see Figure 1).

We analyzed here which kind of geometrical ideas were used to conjecture the solution to Heron's problem, when it was posed to future primary teachers.

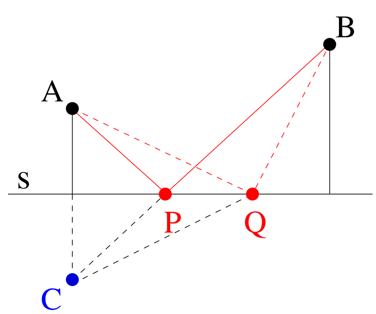


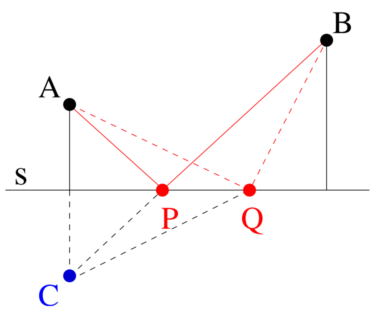
Figure 1:

2 Drawing of visual diagrams to conjecture the solution

Heron's problem was proposed in a primary teacher training course on mathematics. It was their first year at the university, and there were approximately fifty people in attendance. The problem was proposed to the students, who discuss the solution in groups. Afterwards, different approaches were discussed in the whole class.

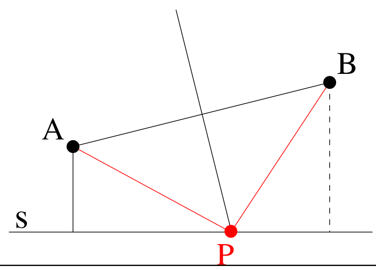
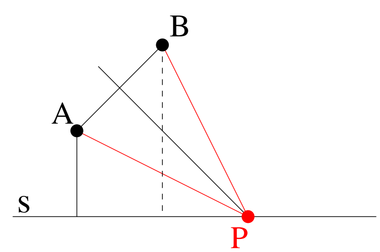
When trying to solve the problem, five conjectures arose, some of them depending on the refutation of others. One of the groups explained conjecture 1 in Table 1 and presented it as a solution of the problem. Afterwards, the whole class were invited to share diverse approaches to the problem, and other conjectures were discussed. These conjectures are summarized in Table 1.

Table 1: Geometric constructions to visualize a solution of Heron's problem.

<table>
<tr><th>Conjecture</th><th></th><th>Proof or counterexample</th></tr>
<tr><td>1. Construction suggested by Heron. P is the intersection between s and the line joining B and the symmetric point to A with respect to the line s</td><td>
</td><td>For any other point $Q \in s$, it must be proved that $\overline{AQB} > \overline{APB}$. Since

$$\begin{aligned} AP &= CP\,, \\ AQ &= CQ\,, \\ AP+PB &= CB\,, \quad \text{and} \\ AQ+QB &= CQ+QB\,, \end{aligned}$$

we deduce from the triangle inequality for $\triangle CBQ$ that

$$\begin{aligned} AP+PB &= CB \\ < CQ+QB &= AQ+QB \end{aligned}$$</td></tr>
<tr><td>2. P is the intersection between s and the median of $\overline{AB}$.</td><td>
</td><td>
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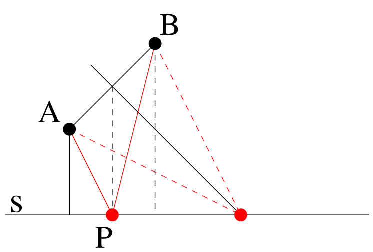
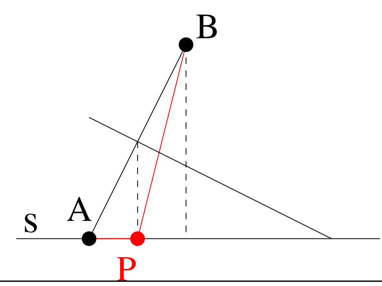
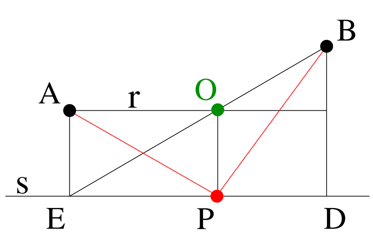
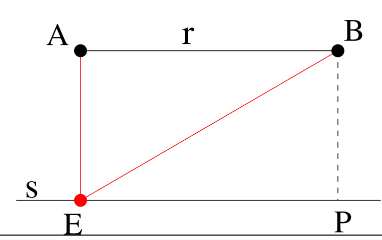
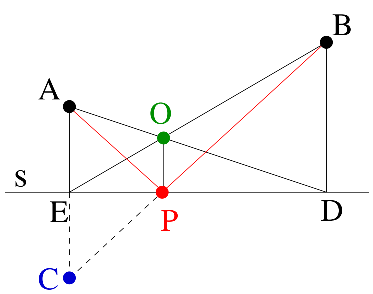
<tr><td>3. P is the intersection between s and its perpendicular through the mid-point of $\overline{AB}$.</td><td>
</td><td>
</td></tr>
<tr><td>4. r is the perpendicular to $\overline{BD}$ through A; O the intersection of r and $\overline{BE}$; and P the intersection between s and its perpendicular through O.</td><td>
</td><td>
</td></tr>
</table>

Table 1: Geometric constructions to visualize a solution of Heron's problem.

Conjecture		Proof or counterexample
5. O is the intersection between $\overline{AD}$ and $\overline{BE}$. P is the intersection between s and its perpendicular through O.		This conjecture gives a correct solution to the problem. Since $\frac{AE}{OP} = \frac{ED}{PD}$ y $\frac{BD}{OP} = \frac{ED}{EP}$, we deduce that $AE \cdot PD = BD \cdot EP$ $AE : EP = BD : PD$ Thus $\triangle AEP$ and $\triangle BDP$ are similar, and $\angle APE = \angle BPD$.

All of them look for the point P by intersecting lines drawn from the line s and the given points A and B. All of the students assumed implicitly that P must lie between the points E y D obtained by intersecting s and the lines perpendicular to s through A and B, respectively.

Attending the use of conceptual procedures in the constructions of visual diagrams to conjecture a solution, these have been firstly classified into two categories:

C1. Constructions based on previous knowledge -conceptual, iconic or procedural- or on the refutation of a previous conjecture.

The first three conjectures in Table 1 follow from this kind of construction. The first one reduces the problem to the consideration of joining two points by a straight line. In this case, the concept of minimal path is embedded in both the problem and the diagram, and those who made this conjecture had the insight to apply their knowledge that the shortest distance between two points is a straight line, but to do so is not trivial, as other knowledge, related to the preservation of distance under reflection, must come into play [15].

The second one determines P as the point of intersection between the perpendicular bisector of $\overline{AB}$ and s. The consideration of P as lying on the perpendicular bisector between A and B allows one to *balance* the distance between the two drawn segments. Students implicitly assign to its points the property of being at the same distance to the extremes of the segment, and this makes this conjecture a good one to solve the problem, illustrating an interesting confusion that occurs in solving such problems: equal distances are often confused with shortest distances.

The refutation of this conjecture, using a new construction where P does not lie on the segment, induces a new conjecture. This new conjecture depends on the previous one, which affirms that P lies on $\overline{AB}$ (Conjecture 3), and could be based on the fact that the mid point of AB is conserved under vertical projection. This new construction permits them to minimize one part (AP) of the total path APB, and is, according to [14], an attempt to control the process of producing a more and more satisfactory drawing. But the solution is also easily refuted by a counter example in which the horizontal distance between A and B is small compared to the vertical distance between them.

The two remaining conjectures were considered independent from the later two, as they arose in different groups after they said they have no idea how to find P.

C2. Spontaneous constructions

Constructions of this kind apparently do not follow any procedure or conceptual strategy, except to obtain a point P between the extremes E y D (Conjectures 4 and 5) [1]. They assume that we expect from them a geometrical diagram leading to P, and produce it by simply making lines starting from the given points, but they are not able to explain why they are doing it so. Although both drawings in conjectures 4 and 5 are created in the same "reckless" way, the fourth is very easy to refute with a simple counterexample; the last conjecture, however, leads to a correct solution for the problem, and generated a long discussion involving the whole class.

When trying to refute the last conjecture by a counter example, all new diagrams proved useless. The students then started to check that *in all cases*, the point P and the one obtained using Conjecture 1 are the same. But the problem is that this new diagram does not permit them to visualize the point P produced by conjecture 5 as a solution of the problem, and even once its validity have been proved, the logical argument involving similar triangles is not significative enough.

3 Didactical approach to conjecture

The analysis of the situation described above led us to four questions addressing what in our view are important demands for research on conjecture, arising directly from the teaching practice:

1. *Conjecturing and proving*

   For the students working on this problem, the use of geometrical drawings to solve the problem is easy, as opposed to giving a proof. Once they have guessed Conjecture 1 (Table 1), the conflict is not that they do not know *how* to prove it, but to understand that they *need* a proof. The construction is considered in itself as a solution of the problem and no symbolical proof is required. For many students, this construction remains a proof, at least until they construct a more elaborated concept [11]. But this is not the case when considering the fifth diagram.

   The formulation of the conjecture, or *empirical* solution [10], needs to offer a key for a second phase involving the *meaning* of a proof in order to be effective. As a consequence, some research linking types of conjectures and proof is needed, emphasizing the more systematic and deductive side of mathematics, and having in mind that

[1] To see the counterexample in 4, consider the locus of all points X in the plane such that $AX+BX = AE+EB$, which is an ellipse through E and D with focal points in A and B. The segment ED is entirely contained in that ellipse, and therefore $AE + ED > AY + BY$ for any point Y in ED, different from E and D.

usually students, and many teachers, look at Mathematics from a more inductive and experimental point of view. Making explicit some nexus between problems, conjectures and proof could generate a proper way to make proof significative for the students.

2. *Types and elements of conjectures*

   Conjecture 5 offers them no clues about its validity, as the first one did. While in the first case the construction is enough for the students to visualize the solution of the problem ("there is nothing to prove –said one of the students–, because the shortest path between two points is a straight line"), this is not the case with the fifth diagram. A priori, spontaneous constructions offer no clue to find the solution. According to [10], even though the use of different representations is a key to progress in problem solving, geometrical representations and their continuous use do not yield, by themselves, the process towards the solution. This last construction instills uncertainty in the students, as there are no elements that enable them to recognize the problem. This generally happens with spontaneous constructions. Students should analyze their geometrical constructions in terms of the given problem in order to be able to use them for solving the problem. Some investigations argued that when producing a drawing, students try to reach harmony between figural and conceptual aspects [14]. However, this does not seem to be so in the case analysed here, as there are some spontaneous constructions which lack this harmony.

   For our practical purpose of analyzing this particular experience from the perspective of teaching practice, the distinction between spontaneous and non-spontaneous has been adequate and productive enough. This is not the case when we want to address a more complete frame to conduct practical experiences involving guessing. This point calls then for research a) characterizing types of conjectures, and b) identifying through categories the cognitive processes involved in each type. This call addresses the more inductive and experimental side of Mathematics, and it is specially relevant for teaching practice, as the students should also learn about effective guessing.

3. *Creating knowledge by linking conjectures*

   Training of visualization or the use of geometrical diagrams to conjecture the solution of a problem should consider the analysis of spontaneous constructions, as these arose frequently during our investigation. The act of visualising and producing new diagrams has an important contextual reference. Here, it is the continuous creation and refutation of conjectures which promotes the creation of new ones. It is in this sense that we are not able to extend these conclusions to other groups and individuals.

   On the one hand, the lack of conceptual knowledge prevents the students from knowing if they are really constructing a satisfactory drawing or not. On the other, the students also need strategies of interpreting their spontaneous constructions in terms of the statement of the problem in order to understand the role of the drawing in the problem solving process.

   At CERME 4 in Sant Feliu, David Reid discussed the problem with us in the working group on proof, and offered us the following possible link between Conjectures 2 to 5 in an hypothetical situation:

   "The diagram created in Conjecture 2 (Table 1) in a context of reasoning suggests the next conjecture (conjecture 3). The problem it reveals is that the point P should stay on the segment ED that is the projection of AB onto s. This constraint, combined with the earlier consideration of trying to equalise distances, suggests that the midpoint of ED, or alternately the projection of the midpoint of AB onto s, is the point P. This new construction also accounts for the known special case. If A is on s, however, a new counterexample is produced (counterexample to Conjecture 3). This diagram shows clearly the correct solution in another specific case: When A is on s then the shortest path is AB itself. The emphasis shifts to defining P is such a way that it can be seen as a continuous transformation from this initial situation. As A moves up, P must move to the right. The segment EB provides a mechanism to produce this motion: A is projected horizontally onto EB to the point O and then O is projected vertically onto s to the point P (Conjecture 4). Again a counter-example is not hard to find (see Table), because this construction does not work in the original special case, when EA=DB.

Combining the two special cases (and the mirror image of the second) suggests Conjecture 5: O is the intersection between AD and BE. P is the intersection between s and its perpendicular through O" [15].

Then, it seems plausible to think that a sequence of conjectures, each based on a single special case –or two special cases, as in the last one– a conjecture is produced that turns out to be correct and it provides a few clues for the proving process, although does not produce an instant proof (as with conjecture 1). How to bring this process of linking conjectures nearer to the students seems to be very important in order to improve their knowledge about guessing, as well as to let the teachers know about the importance of making explicit these plausible links. This would help to conceptualize an otherwise *spontaneous* and often meaningless conjecture.

4. *Problems and conjectures*

Heron's problem has been especially interesting to analyse because it requires an approach via a geometrical construction to conjecture its solution. The problem is also rich enough to give rise to many conjectures. When the students are able to create and interpret properly those constructions, they are often in a good position to conjecture the right solution or to downright solve the problem properly. But not all problems have the same potential to enable the students guessing, as not all problems are equally interesting to learn about proving.

From a very concrete perspective of an in-service teacher, an expository material including commented examples of problems to make understandable types and elements of conjectures would be desirable. Such a material should give them keys to evaluate problems from this point of view, understand and re-create their practice, as well as permit them to question research on the basis of their own practice.

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