NATURAL DEDUCTION IN PREDICATE CALCULUS
A TOOL FOR ANALYSING PROOF
IN A DIDACTIC PERSPECTIVE

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Abstract: *In this paper, we intend to provide theoretical arguments for the importance of taking account in quantification matters while analysing proofs in a didactic perspective, not only at tertiary level, where various research are still available, but also at secondary level and we argue that natural deduction in predicate calculus is a relevant logical reference for this purpose. Following Quine, we emphasize on an example the interest of formalizing mathematical statements in Predicate Calculus in a purpose of conceptual clarification. In a second part of the paper, we give some short insights about the theory of quantification before exposing the system of Copi for natural deduction. The last section is devoted to analysing a proof using the logical tools offered by natural deduction in predicate calculus.*

I. About he importance of quantification in elementary geometry

It is widely recognized that in tertiary mathematical education, quantification matters are central and source of strong difficulties, even for gifted students (Dubinsky & Yparaki, 2000, Selden & Selden, 1995, Epp 2004, Chellougui, 2003) ; but it seems that, in secondary mathematical education, a low interest is paid on quantification matters.

Concerning the French context, in most research in didactic of mathematics, the logical tools used to analyse proofs are referring mainly to propositional calculus. This is the case for Toulmin's model, used for example by Pedemonte (2003) or Hoyles & Küchman (2003). This is also the case for Duval's analysis (Duval, 1991, 1995) widely used in French research about argumentation and proof. In these frames, the focus is on the fundamental inference Modus Ponens. Toulmin (1993) introduces the notion of *warrant*, which is a statement supporting the claim as a consequence of the data. Duval (1995) speaks of "Enoncé-tiers", a statement in the form "if *p*, then *q*", namely a theorem that is already known. Giving such a theorem, once it is stated that, in the considered situation, the conditions required in the antecedent are fulfilled, the conclusion is to be detached. In such frames, the bricks for analysing reasoning are clearly propositions; that means linguistic entities likely to be either true or false. Their main interest is their simplicity and their wide spectrum of application. However, as we have shown it on various examples (Durand-Guerrier 2003a, 2003b), there are a lot of proofs, even in elementary mathematics, that do not fall under this type of analysis. Indeed, quantifications matters are usually involved in mathematical proofs, and not all of them can be absorbed in propositional calculus The ones with which no problem may arise are those classical proofs that require a universal conditional theorem with exactly one free variable: "$\forall x\ T(x) \Rightarrow F(x)$". But of course, even in a Geometry course taught in middle school, you can find statements involving two or more variables and eventually both existential and universal quantifiers. We will now illustrate this point with an example. Let us consider the following statement: "for all three points that do not lie on the same line, there exists a unique circle on which these three points are lying". In order to formalise this statement, three one-place predicates: P (to be a point) ; L (to be a line) ; C (to be a circle) ; a two-place predicate : R (to lie on), and six variables x, y, z t, u, v are required to provide the following formula :

$$(1)\ \Big(\forall x \forall y \forall z P(x) \wedge P(y) \wedge P(z) \wedge \neg\big(\exists t\big(L(t) \wedge R(x,t) \wedge R(y,t) \wedge R(z,t)\big)\big)\Big) \Rightarrow$$
$$\exists u\Big(C(u) \wedge R(x,u) \wedge R(y,u) \wedge R(z,u) \wedge \Big(\forall v\big(\big(C(v) \wedge R(x,v) \wedge R(t,v) \wedge R(z,v)\big) \Rightarrow v = u\big)\Big)\Big)$$

Obviously, this is generally hidden in the mathematic class. The main reason is that most often the figure provides the information expressed in the statement, so that it is no use to have in mind all the condition while solving a classical problem relying on a particular figure. However, if students are supposed to be able to overcome the figure information to deal with general statements, it is worthwhile to keep in mind the complexity of the formula. The syntax of formula (1) reminds us that behind our ordinary simple gestures in mathematics, are hidden many constraints to be taking in account. In this particular case, it becomes apparent when you have to reason in general cases : either it is necessary to control that the considered points do not lie on a same line, or it is necessary to distinguish the particular cases from the general ones. More over, the last part of the formula gives a path to proving uniqueness in such a case. Considering precisely the conditions lead to explicit the difference between a line, a general curve, and a circle, and to emphasize the fact that in Euclidian Geometry "to lie on the same line" for two points is a general result, while for three points or more it is a property that might be satisfied or not, and to continue with the fact that you have the same result replacing "to lie on the same line" by "to lie on the same circle" and "two points" by "three points". Finally, a question such as "what is it for four points" could arise. More generally, it emphasizes the fact that behind symbols, there are objects of various nature and with various properties This is in keeping with Quine' s point of view who claims that the formalisation in Predicate Calculus contribute to conceptual clarification.

II. Difficulties with statements involving quantifiers

We have given in the previous paragraph some arguments that focusing on quantifiers may be useful for teaching mathematics, even at secondary school. But, since the late seventies, after the disaster of the so called " Réforme des mathématiques modernes" in France, logical considerations have been thrown out of programs and progressively from mathematics classroom. In a French famous book addressed to mathematical prospective teachers, Glaeser (1973) notes that quantification matters and consideration about symbol's logical status overcome the purpose of its book, due to the subtlety of distinction between individual and variable.

A way to escape these questions is to work with generic elements, so that quantifiers and variables disappear to benefit of individuals. This is the main way we do in secondary mathematics class in France, following the Euclidian tradition. In this respect, the theoretical frame proposed by Duval (1995) to analyse proofs fit with this ordinary practise. However, as we have shown elsewhere, this may lead on the one hand to deep misunderstanding in statements, particularly those involving implication (Durand-Guerrier 2003a) or negation (Durand-Guerrier & Ben Kilani, 2004), and on the other hand to invalid proofs (Durand-Guerrier & Arsac, 2003). In the last case, it happens that even tertiary mathematical teachers do not recognize the logical error due to an incorrect handling of quantifiers. It can also occur that a well-founded doubt about validity appears while the proof is actually valid (Arsac & Durand-Guerrier, 2000). Our own research corroborates results from Selden & Selden (1995) who claim that unpacking the logic of a mathematical statement is necessary for advanced mathematics, and provide empirical data showing that doing this is very difficult for most students. Chellougui (2004) shows clearly, on the example of the notion of supremum, that difficulties with quantification matters at university are underestimated in French textbooks and, in a Tunisian context rather similar in this respect with the French one, in courses provided to students. Bloch (2000) had already shown that difficulties with quantification matters could arise when dealing with the notion of upper bound. As a consequence, far from contributing to conceptual clarification, formalisation in predicate calculus leads to opacity for definitions and erratic use in proofs. Nevertheless, it appears, in semi-directive interviews that, as soon as there are no more quantifiers involved, students recover their ability to prove.

III. Presentation of the natural deduction in *Symbolic Logic* by Copi.

III.1. Some brief insights about the theory of quantification

Although Aristotle was already deeply concerned with quantification matters, it takes a very long time before a sound quantification theory could be established. According with most authors, Frege (1879) is the first logician who established it on a secure foundation ; he was followed by Russel (1903) and many others who developed it all along the first half-twentieth century. Quine (1902-2002), the famous American logician and philosopher defended all along his life the importance and the relevance of predicate calculus system and its methods for sciences in general and mathematics in particular (Quine, 1950, 1987). In his widely used textbook of formal modern logic (Quine, 1950, fourth edition 1982), two parts apart from four are devoted to quantification - Part 2: General terms and quantifiers; Part 3: General theory of quantification -. A large treatment of proof for logical validity is provided, referring in particular to what he calls the main method, consisting in proof for inconsistency, namely *reductio ad absurdum,* and the dual method for direct proof for logically valid implication. As an introduction of the main method, he wrote :

" We turn now to a proof procedure that will be found to be complete: adequate to establishing the validity of any quantificational schema and hence also to establishing any implication[1] and any inconsistency." (Quine, 1982, p.190)

It is important to point this at the very beginning of the exposition because, in this method, a rule relying on a logically invalid quantificational schema is introduced; and so it is for the dual of the main method for direct proof that he presents further and which inherits of the completeness of the main method. The direct method is "of a type that is known as natural deduction and stems, in its broadest outlines, from Gentzen and Jaskowski (1934)" (ibid. p.244). Many versions of natural deduction can be found in the literature; all of them follow the same purpose: to remain as near as possible of the way mathematicians reason. What we learn from history is that quantification is much more complex than it is generally thought and that the main purpose of the promoters of predicate calculus is the control of validity, especially in mathematic. In this respect, although these methods have been elaborated for controlling validity in Predicate calculus itself, and opposite with other methods for validity, natural deduction systems provide relevant tools for controlling validity of mathematical proofs[2].

II1.2. General features of natural deduction systems

The main interest of natural deduction in a didactic perspective lies in rules for elimination and introduction of both propositional connectors and quantifiers. Elimination for implication is the well-known inference rule called Modus Ponens. Introduction of implication is called by Quine "Condizionalisation" :

"[The rule of conditionalization] consisted in showing that whenever one statement could be deduced from another in the concerned system, the conditional formed of the two statements could also be proved as a theorem by the original rules of that system"(Quine, 1982, p.244)

It is easy to recognize here the classical way to prove a conditional statement in mathematics, in case both antecedent and consequent are propositions. It is related with the theorem of deduction established separately by Herbrand and Tarski and it plays a fundamental role in mathematical proofs. However, it is rather remarkable that the importance of this rule is generally not emphasize in didactic research about reasoning, the focus being preferably pointed on Modus Ponens. The same phenomenon is to be seen concerning the rules for elimination and introduction of quantifiers that are nearly never considered in mathematics classroom, even at the university, being replaced most often by informal reasoning rules (Durand-Guerrier & Arsac 2003)[3]. There are four rules concerning quantifiers ; two of them are relying

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[1] In this textbook, Quine considers that the term "implication" must be kept for the logically valid conditional (for some development, see Durand-Guerrier, 2003, pp.9-10)

[2] For examples, see Arsac & Durand-Guerrier (2000); Durand-Guerrier & Arsac (2003) and Durand-Guerrier & Arsac to appear in *Educational Studies in Mathematics*.

[3] And for a reference in English, Durand-Guerrier & Arsac to appear in *Educational Studies in Mathematics*

on logically invalid quantificational schema so that restrictions are required to insure validity. The main differences in the various systems concern these restrictions. Quine version is very complete and brightly exposed, but it is rather technical and spread in several different chapters. For this reason, we prefer presenting the version of Copi (1954, second edition 1965) that is exposed in a more compact way, and thus most easy to summarise.

IV.2 Quantification rules in the Symbolic Logic by Copi (1954, 1965)

Symbolic logic by Copi is a first textbook written to serve to undergraduate and graduate students, in which, as it is the case in Quine (1950), quantification matters are widely developed. We refer for our presentation to the second edition (Copi, 1965), in which the quantification rules are presented twice. In this paper, we present only the preliminary version, that we will complete by some specific restrictions needed for preserving validity.

The first rule of inference concerns elimination of the universal quantifier ; it is called Universal Instantiation (U.I.) and states that :

" (...) any substitution instance of a propositional function can validly be inferred from the universal quantification. We can express this rule symbolically

$\dfrac{(x)\Phi x}{\therefore \Phi v}$ (where v is any individual symbol)" (Copi, 1982, p.50)

This rule relies on the valid schema $(x)\Phi(x) \Rightarrow \Phi(y)$ and expresses that "*what is true for all is true for any*".

The second rule is the dual of the first one and concerns the introduction of the universal quantifier ; it is called Universal Generalisation (U.G.) and states that :

"(...) the universal quantification of a proposition can validly be inferred from a substitution instance with respect with the symbol y. Our second expression for this quantification rule is

$\dfrac{\Phi y}{\therefore (x)\Phi x}$ (where y denotes any arbitrarily selected individual)" (ibid., p.51)

This rule relies on the invalid schema $\Phi(y) \Rightarrow (x)\Phi(x)$. For this reason, it necessitates a restriction; you must be sure that no assumption other than the property expressed by Φ has been done. Obviously, it is build "by analogy with a fairly standard mathematical practice"(ibid., p.50)

The third rule concerns the introduction of the existential quantifier; it is called Existential Generalisation (E.G.) and states that :

"(...) the existential quantification of a propositional function can be validly inferred from any substitution instance of that propositional function ; (...) Its symbolic formulation is :

$\dfrac{\Phi v}{\therefore \exists x\Phi x}$ (where v is any individual symbol)" (ibid., p.52)

The fourth rule is the more delicate to use. It concerns the elimination of the existential quantifier and states that :

"(..) from the existential quantification of a propositional function we may validly infer the truth of its substitution instance with respect to an individual constant which has no prior occurrence in that context. The new rule may be written as

$$\frac{\exists x \Phi x}{\therefore \Phi v} \text{(where v is an individual constant having no prior occurrence in the context)}$$

(Ibid. p. 52).

As the second rule, this one relies on an invalid schema, namely $\exists x \Phi(x) \Rightarrow \Phi(y)$. The restriction is here very strong and many errors are due to forgetting it, particularly when, in a proof, an existential instantiation follows a universal instantiation[4]. We have shown (Durand-Guerrier & Arsac, 2003, 2005 to appear) that this restriction is closely related with the dependence rule. A consequence of these restrictions is that you must not only have a control for each step of the proof, but also have a control on the global proof. In particular, it is not possible to make a universal generalisation on an individual introduced by an existential instantiation.

IV. Analyse of a geometrical proof

We come back now to the theorem introduced in the first paragraph: "For all three points that do not lie on the same line, there exists a unique circle on which these three points are lying".

A fairly classical proof of this theorem for pupils grade eight can be written in the following form.

Proof: Let A, B and C be any three points not relying on the same line. Let us consider Δ_1 the mediator of the line segment [AB] and Δ_2 the mediator of the line segment [AC]. As the lines (AB) and (AC) are not parallel, then the two mediators are secant; O denotes their intercept. As O is on the mediator of [AB], OA=OB; for an analogous reason, OA=OC. Conclusion, B and C are on the circle, say Γ, whose centre is O, and whose radius is OA. As a circle is perfectly determined by its centre and its radius, this circle is unique.

VI. First analyse of the structure of the proof

First of all, it is to notice that the first assumption is a universal instantiation of the antecedent of conditional that remains implicit in the given formulation of the theorem. Indeed, the formulation is given with a bound quantifier, that restricts the scope of the universal quantifier to those elements that satisfy the required property. But of course, it is to prove a conditional, namely: "For all three points, if they do not lie on the same line, then there exist a unique circle on which these three points are lying".

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[4] See for example Bagni (2005)

The general structure of the theorem to prove is (1) '$\forall x \forall y \forall z(\Phi(x,y,z) \Rightarrow \Psi(x,y,z))$'. and the macro structure of the proof compounds three steps :

1.*To apply Universal Instantiation* to the given statement to get the propositional conditional (2) '$\Phi(a,b,c) \Rightarrow \Psi(a,b,c)$'.This first step remains implicit as it is generally the case in mathematics. The corresponding statement will be "Given three points A, B and C, if they do not lie on the same line, then there exists a unique circle on which these three points are lying".

2.*To prove by "conditionalization"* the statement derived by universal instantiation. In order to do this, an auxiliary premise is introduced. It is a derivation in the proof. Following, for example, Hofstader (1979), we indicate its debut and its end by two brackets :

$[\Phi(a,b,c)$
Mathematical treatment
$\Psi(a,b,c)]$

(2) $\Phi(a,b,c) \Rightarrow \Psi(a,b,c)$ *Introduction of implication*

The brackets indicate that the statement on the first line (the antecedent of the conditional to prove) is introduced as an auxiliary premise, i.e. in this context, it is not assumed as a true statement. As a consequence, the statement on the last line before the bracket (the consequent of the conditional to prove) is not a true statement in the context. So, according with us, writing "conclusion" before this last statement is not relevant. Indeed, the proof is not over! The conclusion of this step is the conditional statement (2) formed by the statements on the first line and the last line in this order.

3.*To apply Universal Generalisation* to statement (2) in order to infer statement (1), that is the statement to prove.

In France, in mathematics classroom, it is rather common to consider only the step 2. More over, in exercises, generally, only the part of the proof that we have written in brackets is considered, without making clear that an auxiliary premise is considered. As an example: (*A,B,C,D*) is a convex quadrilateral ; *I, J, K, L* are respectively the midpoints of its sides [*AB*], [*BC*], [*CD*], [*DA*]. Prove that *(I,J,K,L)* is a parallelogram. Once more, what is to prove is the conditional statement. May be you could think that the status of all these statements are obvious, but it is likely that it is not the case for some students. We may wonder if the nearly exclusive focus on the very core of the proof, where indeed the mathematical treatment is done, could be a didactical obstacle for an adequate appropriation of the specificity of mathematical reasoning, that means proving general results by mean of hypothetical-deductive method. Frege (1971) said that mathematicians often avoid to distinguish between proving $\mathrm{A} \Rightarrow \mathrm{B}$

and proving B by Modus Ponens on $A \Rightarrow B$ and A, and emphasized the importance of this distinction. This question of determining if this practise is a possible didactic obstacle remains open and requires further empirical research. Some argument supporting this thesis is thus provided by Mathematical Induction. Mathematical Induction is introduced in France at grade eleven; it is typically a method of proof that requires clarifying both *conditionalization* and *Universal Generalisation* and it is well known that many students feel strong difficulties to capture the signification of this method.

As it must be clear, the macrostructure that we describe in this paragraph is rather independent of the content of the considered statement. We considered a very general level of syntax, shared by most of mathematical statement, in order to focus to the incompleteness of the proof usually provided in class. However, as we have seen in the first paragraph, the deep structure of the statement is more complex. And the simplicity of the proof we provide is deceptive. Indeed, there are many elements involved in the proof that do not appear, especially those theorems which assert existence under conditions. To illustrate this point of view, we provide in next paragraph an analyse of a short excerpt of the core of the considered proof in Copi's system.

V.2. Mathematical and logical features in the core of the proof.

We consider in this paragraph a small abstract of the proof in order to examine on the one hand hidden existential instantiation and on the other hand how logical and mathematical considerations work together. At this stage of the proof, we are in the core of the proof in which the auxiliary premise '*AP1 : Points A, B, C do not lie on a same line*' has been introduced, and we consider two intermediate conclusions ;

'*C1: A, B, C are three different points*', which is an immediate consequent from AP1, and '*C2 : There exists exactly one point lying on both mediators of [AB] and [AC]* '.

While analysing the proof with Copi's system[5], it appears that it is necessary to express premises under the general form, which is not necessary the case in the standard format. In our cases, the four following premises are required.

P1: For all point M, N and P, if M and N are different, P lies on the mediator of [MN],if and only if PM=PN ;

P2: For all points M, N and P, if M and N are different and if P lies on the mediator of [MN], then P is different from M and N ;

P3: For all two points M and N, if M and N are different, then there exist a unique circle whose centre is M and whose radius is MN ;

P4: For all M, N and P, if M is different from N, and if MP = MN, then P lies on the circle whose centre is M and whose radius is MN.

An immediate remark is that in all these statements interplay implication, universal quantification and for some of them existential quantification. This is a very general case in Geometry, but generally, it remains implicit because most of the information

[5] We give in appendix the analyse with Copi's system of this excerpt

is visible on the figure and consequently only part of the statement is explicit. For example *P3* is replaced by '*P lies on the mediator of MN if and only if PM = PN*', so that the condition that points M and N ought to be different disappears. As every teacher knows, while dealing with general proof, it is very common that students forget particular cases or existence's conditions. Taking care of these particular cases and existence's conditions requires that the focus be moved from propositions to objects. More precisely, if the mathematical effective arguments are indeed expressed through propositions, it is nevertheless necessary, to apply them in a sound manner, to keep control about the handling of quantifiers.

VI. Conclusion

In this paper we try to show in which respect natural deduction in predicate calculus provides tools for analysing proof by taking in account quantification matters. By the exigency of introducing every symbol used in the proof, distinguishing dummy variables and individual symbols, natural deduction in predicate calculus offers a possibility to control the validity of proof and in certain cases point out implicit that might lead to incorrect proof. By focusing on interplay between propositional connectors and quantification, it provides an extension of the classical tools used in didactic of mathematics to analyse proofs. According with us, this enlightens the fact that many things are silenced in the standard manner, especially those considerations concerning the existence of the objects that are introduced. We have shown that it is relevant for analysing proof in tertiary mathematical education. We make the hypothesis that it is also the case in secondary mathematical education, and we thought that we give in this paper some insights to open a path for further research considering this question. More generally, these considerations show clearly that for us, the logical validity is a prominent criteria for analysing proofs in a didactic perspective, prior from the focus on *proving to convince* or *proving to explain*[6].

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[6] Although we do not underestimate these aspects.

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Appendix

Analyse with Copi's system of an excerpt of the proof studied in section V.

The first seven statements are the premises. Let us remind that AP is the auxiliary premise, antecedent of the conditional to prove, and that C1 and C2are intermediate conclusions, while P1, P2, P3, P4 are theorems.

(1) AP. Points A, B, C do not lie on a same line

(2) C1. A, B, C are three different points

(3) C2. There exists exactly one point lying on both mediators of [AB] and [AC]'

(4) P1. For all point M, N and P, if M and N are different, P lies on the mediator of [MN],if and only if PM=PN ;

(5) P2. For all points M, N and P, if M and N are different and if P lies on the mediator of [MN], then P is different from M and N ;

(6) P3. For all points M and N, if M and N are different, then there exist a unique circle whose centre is M and whose radius is MN ;

(7) P4. For all M, N and P, if M is different from N, and if MP = MN, then P lies on the circle whose centre is M and whose radius is MN.

(8) O lies on both mediators of [AB] and [AC] — *Existential Instantiation on (3)*

(9) O lies on the mediator of [AB] — *Separation on (8)*

(10) if A and B are different, then O lies on the mediator of [AB] if and only if OA = OB — *Universal Instantiation on (4)*

(11) A and B are different — *Separation on (2)*

(12) O lies on the mediator of [AB] if and only if OA=AB — *Modus Ponens on* (10) & (11)

(13) OA=OB — *Modus Ponens on (9) & (12)*

(14) O is different from A and from B — *Universal instantiation followed by Modus Ponens on (9), (11) & (5),*

(15) O is different from A — *Separation on (10)*

(16) There exist a unique circle whose centre is 0 and whose radius is OA. — *Universal instantiation followed by Modus Ponens on (15) &(6)*

(17) Γ is the circle whose centre is 0 and whose radius is OA — *Existential instantiation on (11)*

(18) B lies on the circle Γ whose centre is 0 and whose radius is OA — *Modus Ponens on (11), (13) &* (7)

(19) C lies on the circle Γ whose centre is o and whose radius is OA — *Substitution from (9) to (18)*

(20) A lies on the circle Γ whose centre is O and whose radius is OA — *Substitution in (13), (15), (18).*

*(*21) A, B and C are lying on circle Γ — *Conjunction on (19), (20), (21)*

(22) There exists a circle on which points A, B and C are lying — *Existential generalisation*