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Proof

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WHAT IS PROOF ?

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By David A. Reid

Abstract

This paper reports on the diversity of meanings attached to the words "proof" and "proving" by upper elementary school students and makes connections with similar usages by mathematicians and mathematics educators. Five dimensions of meaning are used to organise the discussion: need, ways of reasoning, social target, form, role in a community. Three needs are identified in the students use of the words "proof" and "proving": verifying, explaining why and explaining how. Two ways of reasoning are identified: inductive and deductive. Social targets include oneself, others, and combinations of the two. Formal contexts and characteristics mentioned by the students include geometry, particular words ("because", "therefore") and use of symbols. Roles include the traditional view of proofs assuring certainty and a more Lakatosian view of proofs as provisional. This diversity in usages by students and by mathematicians and mathematics educators is concluded to be a good thing, but not without pitfalls.

Executive summary

This paper reports on the diversity of meanings attached to the words “proof” and “proving” by upper elementary school students and makes connections with similar usages by mathematicians and mathematics educators. It builds on the work of Godino and Recio (1997) who described several different meanings of “proof” in institutional contexts.

The data was gathered in a grade five classroom where problem solving is emphasised. The students were asked to write and talk about proof on questionnaires and in interviews.

Five dimensions of meaning are used to organise the discussion: The need satisfied by an activity; The ways if reasoning used; The social target of the activity; The form the product of the activity takes; And the role the activity plays in a community. Three needs are identified in the students use of the words “proof” and “proving": verifying, explaining why and explaining how. Two ways of reasoning are identified: inductive and deductive. Social targets included oneself, others, and combinations of the two. Formal contexts and characteristics mentioned by the students include geometry, particular words (“because”, “therefore”) and use of symbols. Roles include the traditional view of proofs assuring certainty and a more Lakatosian view of proofs as provisional. This diversity in usages by students is interpreted as a reasonable development given the advantages of diverse views in a community. This conclusion is also applied to the diversity of usages among mathematicians and mathematics educators. Caution must be employed,

however, to ensure that members of a community are aware of others' meanings and to be clear about ones' own.

Maybe some mathematicians have different ideas of proofs than others (Samantha, May 27, 1999, in reply to the question "What do you think mathematicians would consider a proof?")

It is a widely acknowledged fact that high school and university students do not understand what is meant by "proof" and "proving" (Schoenfeld 1989, Harel & Sowder 1998, Balacheff 1988, Chazan 1993, Fischbein 1982). For researchers examining the learning of proof from a constructivist or similar perspective, an interesting question is "What *do* students understand by the words 'proof' and 'proving'?" This paper reports on the diversity of meanings attached to these words by students in a grade 5 class (ages 10-11) and makes connections with similar usages by mathematicians and mathematics educators.

Godino and Recio (1997) describe several different meanings of proof in different institutional contexts: the foundations of mathematics, professional mathematics, empirical sciences, daily life, and school mathematics. They observe that in the foundations of mathematics, proof must be both formal and deductive, while in professional mathematics they assert that "proofs are deductive but not formal" (p. 316). In the experimental sciences and daily life they see proof as a mixture of deductive, empirical inductive and analogical reasoning. In schools the reasoning used to establish mathematical statements ranges even more widely and can include the voice of authority of the teacher. Another significant difference is that the statements being proved in schools are known to be true in advance. The need for proving them is different than it is for professional mathematicians seeking to prove new statements. By building on the exploration of the diversity of meanings for "prove" and "proving" among grade 5 students, this paper extends the work of Godino and Recio both in identifying elements of proof and proving that vary in different understandings, and also in showing that the understandings within the mathematical and mathematics education communities are not as uniform as Godino and Recio suggest.

In the springs of 1999 and 2000 I spent a week with Vicki Zack exploring the meanings of proof and proving held by the students in her grade 5 class.[1] It turned out that they had a wide range of meanings for "proof" and "proving." One interpretation of this would be that most of these students did not know what "proof" and "proving" mean in mathematics. Another interpretation, and the one adopted in this paper, is that the words "proof" and "proving" have multiple meanings, not only in everyday life, but also in mathematics and mathematics education, and that the students' meanings are comparable to the range of meanings found in the mathematics education literature.

Background

Zack is a teacher-researcher who has been engaged in researching her students' communication in mathematical contexts for several years (Zack 1995, 1997, 1998, 1999ab; Graves & Zack, 1997, 1996). The school is private and non-denominational, with a population that is ethnically, religiously and linguistically mixed. Most students

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[1] This research was carried out with funding from the Social Sciences and Humanities Research Council of Canada grants #410-98-0085, #410-94-1627 and #410-98-0427.

come from English speaking, middle class backgrounds. Non-routine problem solving is central to the mathematics curriculum at all grade levels.

> The school and classroom learning site is a community of practice which Richards (1991) has called inquiry math; it is one in which the children are expected to publicly express their thinking, and engage in mathematical practice characterized by conjecture, argument, and justification (Cobb, Wood, & Yackel, 1993, p. 98). (Graves & Zack, 1996, p. 28)

Mathematics is studied for 45 minutes each day (sometimes extended to 90 minutes), and extended investigations of non-routine problems take up the entire lesson three times a week. There are usually about 25 students enrolled in the class, but mathematics is studied in a half class of 12 or 13. The students are grouped heterogeneously in groups of four or five. When working on a problem they first work in twos and threes, then come together in their groups to compare solutions, and then report to the half-class. (Other episodes and interpretations involving some of the same children can be found in Zack, 1999ab.)

The students were videotaped throughout their group and half-class discussions. In addition, their written work in their “math logs” was photocopied, written responses to questions focused on particular aspects of their activity were collected, and they were interviewed and videotaped reflecting on their past activity. The data produced covers the mathematical activity of all the children in each year’s class, for five years.

The choice of problems and the teaching methods employed are Zack’s. My role as a researcher is limited to ongoing discussions of the events in the classroom, and observations and interviews during one week in May when the school schedule allows for small groups to be removed from the class at almost any time, and for more time overall to be spent on mathematical problem solving.

The quotes in this paper come from interviews, entries in students' math logs and responses to questionnaires completed in May 1999 and 2000. During the weeks May 25-28, 1999 and May 23-26, 2000 I was present in the class, and a particular focus of those weeks was on the students' ideas of proof and proving. During this period the problem they were working on was determining the number of squares (of all sizes) in a *n*by*n* grid of squares. The problem was first posed for the specific case of a 4by4 grid, then for a 5by5 grid, then for 10by10 and 60by60 grids.

The students were not taught any meanings for "proof" or "proving." They had been shown and asked to write "proofs" in the context of geometry, but no connection had been made between that activity and the weekly problem solving activity. Normally when solving problems they were asked to check their answer and "to write what you did in your checking" in their math logs. Beginning with their solutions to the 4by4 problem they were also asked to "Prove your answer is correct." When they considered the other cases and they were again asked to "Prove" their solutions, as well as to write how they "checked" them. There was no discussion of their responses (except for comments written in their logs in a few cases) until May 25, 1999 and May 19, 2000, when they were given copies of the responses of all the members of the class, and asked to "Write about CHECKING and PROVING, and tell us what you think." They were interviewed in small groups about their answers to this question and other aspects of their solutions to the problem over the following days, as well as (in 1999) answering in writing the question “What do you think mathematicians would consider a proof?” on May 27.

Dimensions of proof

The multiplicity of meanings for proof and proving arises from the fact that proof and proving have several dimensions of meaning, and many meanings arise from focus on a single one of these dimensions, or on combinations along these dimensions. An act can be called proving based on (at least) five dimensions of meaning:

1. The **need** (purpose, function) that the act is intended to address. Some values on this dimension are explanation, exploration, and verification. These and other needs to prove have been described by de Villiers (1990).

2. The way(s) of **reasoning** involved in the act: deduction, induction, analogy, etc. Pólya (1954/1990) describes these ways of reasoning and some relationships between them.

3. The **target** of the act: oneself, a peer, a larger community, a teacher. Mason, Burton and Stacey (1985, p.95) identify three stages in justification: “convince yourself, convince a friend, convince an enemy” each requiring a different style.

4. The **form** of the product of the act: two-column proof, semi-formal proof, pre-formal proof. The most common form of proof in North American classrooms is the two-column proof taught in the context of geometry. A possible goal for teaching is the semi-formal style of proof used in mathematics journals (“formal” being reserved for

theorems in formal logic). Blum and Kirsch (1991) describe "pre-formal proofs" as a possible step to semi-formal proofs.

5. The **role** the act plays in a mathematical community. It is generally agreed that proof is central to mathematics, and is a defining characteristic of mathematics. There remains some disagreement on the role of proof in mathematics, between those who see proof as offering certainty, and those (e.g., Lakatos, 1976) who see mathematics as a quasi-empirical study in which theorems are only provisionally true, and all proofs are fallible.

Needs

Proving is verifying your idea. I say checking and proving are the same thing and I don't understand how they could be different. (Ryan, May 25, 1999, replying to the question "Write about checking and proving")

Characterising proving as verifying, checking, justifying, convincing, establishing, etc. is very common in both everyday speech and mathematics education research and was the most common element in the students' definitions of proof and proving. A typical definition given in education literature is that of Sowder and Harel (1998):

> Proving, or justifying, a result involves ascertaining – that is, convincing oneself – and persuading, that is, convincing others. An individuals' proof scheme consists of whatever constitutes ascertaining and persuading for that person. … The word *proof* in *proof scheme* is used in the broader,

> psychological sense of *justification* rather than in the narrower sense of *mathematical proof*. (p. 670, italics in original).

This is the way people use "proof" and "proving" in everyday speech:

Proving to me means to show what you said is right. For example the white guy in Roll of Thunder Hear my Cry *proved that T.J. stole the pistol because the pistol was in T.J.'s house. (Zulan, May 4, 2000, in response to the prompt "Write what proving means to you".)*

Building on this everyday meaning of proving might seem like a reasonable basis for teaching proof. As Sowder and Harel advocate, students could be guided from external proofs (e.g., the word of authorities) to empirical proofs (experiments and examples) and finally to analytic proofs (those acceptable to mathematicians). One problem with this idea is that mathematics is unusual in that only certain ways of reasoning are accepted as proving (see Roles, below). In no other domain of human activity is the move to analytic proofs sensible, or even possible. Motivating analytic proofs in mathematics teaching becomes difficult if proving is characterised as verification (de Villiers, 1990).

Check if your answer is right. Prove why it's right. Why it's right instead of that it's right. (Andy, May 25, 1999, interview)

Another need for proof is the need to explain, to illuminate, to understand. Explaining is often mentioned in current research on proof and proving as an alternative to verification. Bell (1976) pointed out some time ago that "a good proof is expected to convey an insight into *why* the proposition is true" (p. 24, italics in original). This insight has since been repeated by a number of researchers, including Hanna (1989), de Villiers (1990) and Reid (1995b), who have also advocated using the need to explain in schools as a motivator for teachers' proving, and students' learning of proving. One difficulty in this approach is that some types of proof (e.g., proof by mathematical induction) and some forms of proof (e.g., symbolic proofs with a high degree of formality) do not have an explanatory quality for many people (Hanna 1989).

Checking is checking your answer and proving is showing how your answer works. (Ron, May 25, 1999, replying to the question "Write about checking and proving")

Understanding and explaining are usually meant by mathematics educators to refer to questions of *why* something is the case. In school however, procedures are often presented as being more important than explanations. This leads to a tension between what Skemp (1987, 152 ff) calls relational understanding (knowing why) and procedural understanding (knowing how). School practices can enculturate students into a focus on procedures, or deny opportunities for the development of relational understanding, leading to students' responding to requests to "explain why" by providing step by step descriptions of *how* they came to a result (as described in Reid, 1999).

This focus can also lead to students seeing a proof as nothing but a template to guide specific procedures, as in the case of this university student:

> David: What do you think a proof is for?
>
> Claire: To give guidelines, to umm, to be able umm —
>
> David: What do you mean by guidelines?

Claire: Well I guess — to give you guidelines in working out a problem or something, you have a proof that gives you a direction or gives you a — it gives you a formula that you can plug in any numbers to get where you want to go.

David: Would Pat's proof be useful in that way?

Claire: Yeah Pat's proof would

David: Would your demonstration of the truth of the statement be useful that way?

Claire: Well , what do you mean, by plugging in any numbers I can prove her statement? Yeah.

(from Corbeil & Reid, 1990)

Ways of reasoning

I thought this could be proving because the same pattern occurs again and again. (Marilyn, May 3, 1999, math log entry)

Marilyn's observation of a repeating pattern is one kind of inductive reasoning, or reasoning from examples, experiments, or observations. Inductive reasoning is often described as the *source* of conjectures. Less often it is included in discussions of proof and proving. Examples include Bell's (1976) inclusion of "empirical" arguments as "proof strategies," and Sowder and Harel's (1998) "empirical proof schemes." It is always dangerous to extend discussions of word usage across languages, but Balacheff's (1987, p. 163) description of "empirisme naïf" as a "type de preuve" would also seem to fall into this category. This usage is especially common in the work of those who consider verification to be the need that motivates proving (e.g., Sowder & Harel, 1998). It is also common in the work of those studying argumentation in young children (e.g., Maher, Steencken, & Deming, 1996).

Proof to me is a logical explanation. Since I used logic in my problem my explanation shows the logic and therefore would be the same as my proof. (Walt, April 29, 1999, math log entry)

An examination of Harrison's explanation shows that the logic he is referring to is deductive, and deductive reasoning is taken by many researchers in mathematics education as either the only reasoning that counts as proving (e.g., Reid, 1995a), or at least the "ultimate type" (Sowder and Harel, 1998, p. 673). In this view the focus is on the way of reasoning more that the need that reasoning satisfies, in contrast to those who include inductive reasoning as proving, for whom the need to verify is essential.

Targets

When you are proving you are proving not only to the teacher but to yourself also because you have to prove to you to say to yourself that you could do the problem. (Zina, May 25, 1999 replying to the question "Write about checking and proving")

Proving can be seen as a personal activity, done for one's own benefit. Schoenfeld (1982) expresses this view of proving as something directed inwardly:

> Proving is a means of coming to understand, and of coming to know what understanding is. In trying to prove something new, one is asking what makes it tick; in trying alternative proofs, rejecting them, modifying them,

> one is discovering things about its structure – and solidifying one's knowledge in the process. … The mathematician comes to accept proving as a way (if not *the* way) of thinking, a way of demanding and insuring that he does indeed understand. (p. 168, emphasis in original)

Research based on such a view of proving is likely to concentrate on the reasoning of individuals as opposed to the role of proving in groups, which in turn has implications for the needs that proving might be seen as addressing and the reasoning that might be used.

I think checking and proving are totally different! *Checking it's to make sure for you and the others that your answer is right. Proving it's to prove to the other that you are right (Jeanne, May 19, 2000, in response to the prompt "Write about CHECKING and PROVING and tell us what you think.")*

Proving can be seen as a social process, engaged in to provoke some sort of a reaction from others. Maher and Martino (1996) trace the proving activity of Stephanie from grade 1 to grade 5, and include this statement illustrating her goal in proving: "So I've convinced you that there are only eight?" (p. 194). For Stephanie, and for Maher and Martino, proving is a way of interacting with others. Balacheff (1987) also sees proof as socially defined, "Nous appelons **preuve** une explication acceptée par une communauté donnée à un moment donné." (p. 148, bold in original)[2]. Researchers who concentrate on others as the targets of proving have an advantage in that their object of study is recordable and transcribable. On the other hand, determining what a *community* accepts is often much harder than determining what an individual accepts.

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[2] We call **proof** an explanation accepted by a given community at a given time.

If you did a mathematics problem like our problem solving, mathematicians have to proof to himself that the problem is correct then he proves to other people that it is right. (Zina, May 27, 1999, in reply to the question "What do you think mathematicians would consider a proof?")

Most researchers in mathematics education would acknowledge that proving is both a social and a personal activity, and in fact that there is little sense in limiting the target of proving. This probably applies to the researchers quoted in the preceding two sections, although it is difficult to tell from their work. Being aware of the target of proving is important however, when examining particular cases, as proving done for oneself may differ from proving done for others.

Forms

Proof for me is like geometry. (Jake, May 25, 1999, replying to the question "Write about checking and proving")

Geometry is the context in which proof is often taught, and for a long time it has been put forward as both the best context for the learning of proof, and as not worth learning unless as a context for the learning of proof.

> If demonstrative geometry is not taught in order to enable the pupil to have the satisfaction of proving something, to train him in deductive thinking, to give him the power to prove his own statements, then it is not worth teaching at all. (Reeve, 1930, p. 14.)

The only examples of “proofs” these children had seen were in geometry. “Proof” was not defined at that time, but instead, as is commonly done in teaching, examples were shown to be imitated. One of these examples is shown in Figure 1.

Insert Figure 1 about here

Given this prior experience with proof it is perhaps surprising that more of the children did not make the connection with geometry (Only three others in Jake’s class mentioned geometry). It is also interesting that no one other than Jake mentioned geometry when asked “Write about checking and proving.” The other three were responding to the question “What do you think mathematicians would consider a proof?”, which suggests that their experiences with proof in geometry were disconnected from their own ideas about what “proof” might mean.

Recently a move away from the teaching of geometry as a context for proof has been advocated, as it is blamed for students’ seeing proof as relevant *only* to doing geometry, and as restricted to the formats used in that context. Documents such as the *Standards* (NCTM, 1989, 2000) advocate the teaching of proof across all areas of mathematics, and the use of other forms of proof in addition to those that are traditional to geometry.

…the thing with the therefore, because (Mona, May 27, 1999, in reply to the question “What do you think mathematicians would consider a proof?”)

The geometry proofs that Zack’s students produced had a particular style, that Jacqueline has summed up concisely. Proofs in mathematics journals also follow particular styles (although each area of mathematics has its own style). They all have a *semi-formal* degree of formality, in between the rough *pre-formal* notes of a work in progress and the explicitness of a work of formal logic or computer programming in which every step is written. As can be seen in the sample proof Zack's students were given (see Figure 1), they first worked out the answer, which amounts to a pre-formal proof, full of unstated assumptions and without any clues as to the logical connection between steps. They then produce a semi-formal proof in the particular style of the class. They had to state the prior knowledge they were using (marked with "Because") followed by the statements they could conclude from that prior knowledge (marked with "Therefore"). Blum and Kirsch (1991) advocate the use of pre-formal proofs in schools as a stage toward semi-formal proofs which seems sensible given that pre-formal proving seems to arise quite naturally in children (Anderson, Chinn, Chang, Waggoner and Yi, 1997).

Proving your answer using a mathematical symbol (Fiona, May 27, 1999, in reply to the question "What do you think mathematicians would consider a proof?")

Symbols are often seen by students as being important to mathematics, and this attitude extends also to some teachers. For example, a college teacher interviewed by Roberts (1999) felt that the following proof that the sum of the interior angles of a Euclidean plane triangle is 180 degrees "uses no mathematical concepts" (p.81), perhaps because it uses no symbols and is not in a two-column format.

> If you walk all the way around the edge of the triangle [Figure 2], you end up facing the way you began. You must have turned a total of 360°. You can see that each exterior angle when added to the interior angle must give 180° because they make a straight line.

This makes a total of 540°. 540° – 360° = 180°

Insert Figure 2 about here

Requiring that proofs include symbols is not widely done in mathematics education, however, but it is a concern that for many students it is symbols that make something a proof.

Roles

Mathematicians use the term mathematical proof to explain that if you were to want to prove your math answer you would have to use a mathematical proof (Aline, May 27, 1999, in reply to the question "What do you think mathematicians would consider a proof?")

Alix is quite right: proving is what mathematicians are doing when they say they are proving. "A proof becomes a proof after the social act of 'accepting it as a proof'." (Manin, 1977, p.48) Mathematicians, like the members of other disciplines, define their discipline by the kinds of questions they ask and the kind of reasoning they accept in answers. This choice is based not on hard and fast rules, but on implicit, socially negotiated criteria that become internalised as preferences as a person becomes a mathematician. Maturana (1988) refers to such internalised preferences as "emotional orientations" as they are based not on rational criteria, but instead are the substratum on which what counts as "rational criteria" is decided. The object of mathematics teaching is sometimes said to be to have students act like mathematicians. To do so would amount to adopting the mathematical emotional orientation, for which we do not yet have an adequate description. One step in describing the mathematical emotional orientation is considering the role of proof in mathematics.

A logical explanation that shows that you are right and there is no way around it even if you turn it upside down. (Walt, May 27, 1999, in reply to the question "What do you think mathematicians would consider a proof?")

One version of the role of proof in mathematics is that it establishes the absolute certainty of theorems. It can do this because deductive reasoning itself is certain. If the premises of an argument are true, then the conclusion will be true, as long as the reasoning used is purely deductive. Because deductive proof is the only way to verify a theorem, mathematical truths are absolutely true, unlike the truths in other disciplines which are based on empirical or authoritarian methods.

This is the role of proof in mathematics that has been put forward by mathematics educators throughout history.

> Of all those who have already searched for truth in the sciences, only the mathematicians were able to find demonstrations, that is, certain and evident reasons. (Descartes, 1637/1993, p.11)

> The purpose of geometry is to make clear to the pupil the meaning of demonstration, the meaning of mathematical precision, and the pleasure of discovering absolute truth. (Reeve, 1930, p. 14)

> All students, especially the college intending, should learn that deductive reasoning is the method by which the validity of a mathematical assertion is finally established. (NCTM, 1989, p. 143)

Holders of this view necessarily assigned certain values along the other dimensions of the meanings of proof. The need to verify is paramount, as is deductive reasoning. Semi-formal proofs addressed to the community of mathematicians are emphasised, and proofs (the product of proving) are more important then the process of proving itself.

I think you can check and prove a wrong answer if you think it's right. Proving is thinking your answer is correct until someone proves it wrong. (Marilyn, May 25, 1999, replying to the question "Write about checking and proving")

Marilyn's answer seems to come from an alternative vision of the role of proof in mathematics. Lakatos (1976) puts forward the idea that the role of proof is to provide a target for criticism, not to establish the truth of a statement. Once a proof has been suggested, the search is on for counter-examples. When a counter-example is discovered, it will be found either to contradict some step in the proof (in which case the proof is revised and the process resumes) or to contradict the statement that was meant to be proved, without contradicting any step of the proof. In this case there are two paths for future investigation: revising the statement to avoid the counter-example, and trying to determine what statement the proof might actually prove, as it can not prove the original statement. This cycle of proofs and refutations is endless, although in practice it might happen that no counter-examples are found and the statement and its proof might be left on the books, assumed to be true.

This is a very different version of the role of proof than the previous one. Instead of a theorem being absolutely true, it is merely thought to be true, until someone disproves it. Instead of the proof transmitting truth from the premises to the conclusion, it retransmits falsity from a statement to its antecedents when a counter-example arises. This idea is becoming increasingly popular with mathematicians with a philosophical bent (e.g., Davis & Hersh, 1986) and has had some influence on mathematics educators (e.g., Balacheff, 1987; Reid, 1995a).

This view of the role of proving is open to multiple needs to motivate proving, and to a range of forms of proofs. The targets of proofs include both individuals and the larger community, and the focus is on the process of proving more then the proofs produced. Deductive reasoning is essential to the view of proof, but the involvement of other ways of reasoning in support of the proving process is acknowledged.

Conclusion

From the above it will be clear that Zack's students said and wrote things about proof and proving that suggest they have a wide range of understandings. They differ in the needs they expect proving to address, the audiences they see proofs as targeted towards, to kinds of reasoning they accept as proving, the forms they expect proofs to take, and perhaps even the role they see for proof in mathematics. It should also be clear that mathematics educators have a wide range of understandings of proof and proving, differing across the same dimensions.

Given the divergence in the students responses, we might ask if it is a problem that Zack's students gave many different answers to questions about the nature of proof and proving. Should she try to clarify their understandings so that they agree on one description? Which one?

We might also ask if it is a problem that the members of the mathematics education community give different answers. Should we come to agreement among ourselves on the "real" meaning of proof and proving? Which one?

As a community Zack's students are aware that their meanings for proof and proving are different. They feel no tension about this however. The meanings they have are sufficient for their purposes. And the diversity of meanings in the class means that in a given situation a viable description of proof or proving is likely to be present. They have achieved something analogous to the diversity within a species that means that in a shifting environment, some individuals will be able to cope, and in an interdependent community that diversity benefits everyone.

As researchers the mathematics education community also works better by allowing for a range of understandings of our central concerns. Students' learnings are diverse, and the study of them must make use of diverse perspectives. Phenomena that resist explanation from some perspectives may be more amenable to being explained from others, and so the diversity of perspectives in the mathematics education community benefits all of us.

With these benefits, however, comes a responsibility. A perspective that is different from one's own is easy to reject, and there are some perspectives that ought to be rejected, as useless or harmful to our various aims. In order to sort out useful perspectives from harmful ones we must understand those perspectives. The challenge is to maintain the diversity of perspectives that strengthens the community, without losing the mutual understanding that allows us to communicate and to reject, as a community, those perspectives that are clearly wrong. In the face of this challenge we must strive to make our own perspectives clear, enough that they can be understood by those that do not share them. Labouring under the assumption that we all think alike will only lead to confusion. Acknowledging that we think differently, communicating our differences, and acknowledging the values of those differences provides the basis for a research community that is both flexible and strong.

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Working out my answer:

```
180
-30
---
150

  75
 ---
2)150
  14*
  ---
   10
```

Therefore <R=75°

Writing my proof:

Because the sum of the three angles in any triangle equals 180°
Because its an isosceles triangle, <R=<R
Because one angle equals 30°
Therefore

```
180
-30
---
150
```

Therefore 150÷2=75°
Therefore <R=75°

Figure 1: Sample Proof, March 2, 2000.

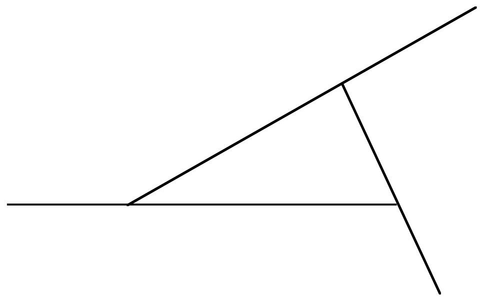

Figure 2

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