RELATIONSHIP BETWEEN BEGINNER TEACHERS IN MATHEMATICS AND THE MATHEMATICAL CONCEPT OF IMPLICATION

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In this paper, we present a didactic analysis of the mathematical concept of implication under three points of view : sets, formal logic, deductive reasoning. For this study, our hypothesis is that most of the difficulties and mistakes, as well in the use of implication as in its understanding, are due to the lack of links in education between those three points of view. This article is in the continuation of those previously published in the acts of PME 26 and PME 28. We present here the analysis of our experimentation's results, that we have not yet shown.

INTRODUCTION

The implication is an usual object of our everyday life that we use to communicate. Its existence in natural logic leads to confuse it with the mathematical object, which then seems to be a clear object. Yet, even training teachers in mathematics have difficulties related to this concept of implication, especially with regard to necessary and sufficient conditions.

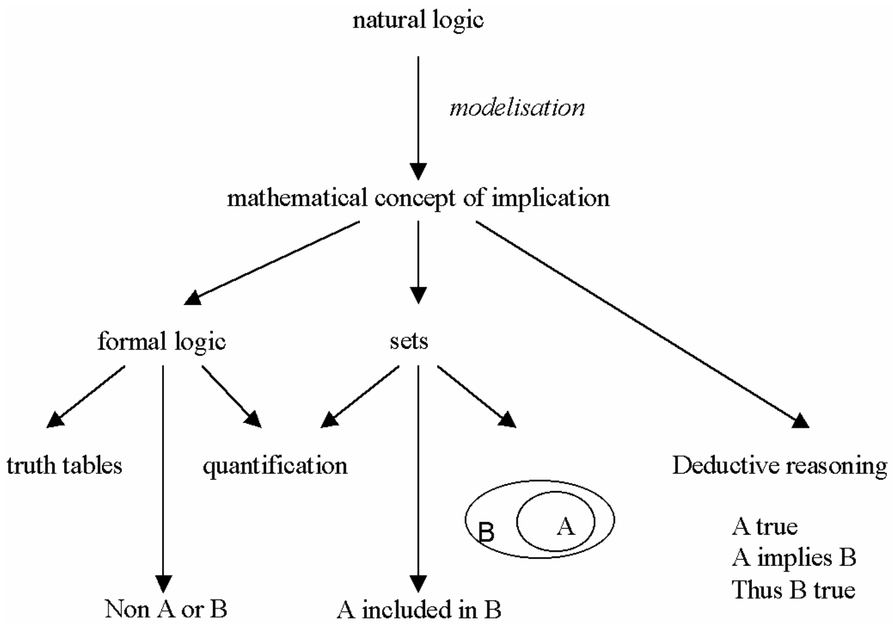
The study we present here is a part of our thesis on the mathematical concept of implication [Deloustal-Jorrand, 2004 c]. Our theorical framework is placed in the theory of french didactics, in particular, we use the tools of Vergnaud's conceptuals fields theory and those of Brousseau's didactical situations theory. Our study is linked to the work of V. Durand-Guerrier [Durand-Guerrier, 2003] on the one hand and of J. and M. Rogalski [J. & M. Rogalski, 2001] on the other hand. V. Durand-Guerrier shows, in particular, the importance of the contingent statements for the comprehension of the implication. J. and M. Rogalski try to define types of structuring of the use of logic when evaluating the truth of an implication with a false premise. We do not forget that the implication is an essential tool for the proof. Yet, we choose here to focus our research on this concept rather than on a the proof in general.

This study follows and supplements those presented at PME 26, PME 28 and ICME 10. We give now results that were lacking previously. We present, first, three points of view on the implication before a mathematical and didactical analysis of one of our problem tested with training teachers, after what we give some results and conclude.

THREE POINTS OF VIEW ON THE IMPLICATION

This paragraph was detailed in our previous research report in PME 26. Yet, we think this part of our research is necessary for the reader to understand the following problem and the aim of our research hypothesis, hence we summarize it here.

The mathematical implication seems to be a model of the natural logic implication we use in our everyday life. Like any model, this mathematical concept is faithful from certain angles to that of natural logic but not from others. This distance between the mathematical concept and the natural one leads to obstacles in the use of the mathematical concept. An epistemological analysis [Deloustal-Jorrand, 2000] enabled us to distinguish three points of view on the implication : formal logic point of view, deductive reasoning point of view, sets point of view.



Of course, these three points of view are linked and their intersections are not empty. We develop here neither the formal logic point of view (for example truth tables or formal writing of the implication) nor the "deductive reasoning" for which one can refer to Duval [Duval, 1993, p 44]. In the “deductive reasoning”, the implication object is used only as a tool. However in French secondary education, where this point of view is the only one, it often acts as a definition for the implication.

Generally speaking, having a sets point of view, means to consider that properties define sets of objects: to each property corresponds a set, the set of the objects which satisfy this property. The sets point of view on the implication can then be expressed as follows: in the set $\boldsymbol{E}$, if $\boldsymbol{A}$ and $\boldsymbol{B}$ are respectively the set of objects satisfying the property A and the set of objects satisfying the property B. Then, the implication of B by A (i.e. $A \Rightarrow B$) is satisfied by all the objects of the set $\boldsymbol{E}$ excluded those which are in ***A*** without being in ***B***, i.e. by all the objects located in the area shaded here after. In the particular case with $A \subset B$, all the objects in E satisfy the implication.

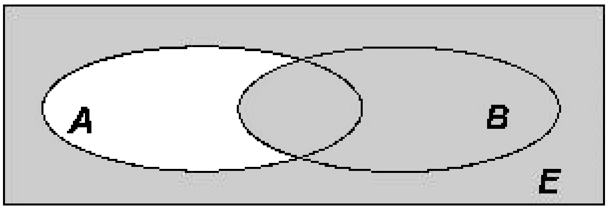


Figure 1

RESEARCH HYPOTHESIS

The experiments carried out for several years, within the framework of our research, showed that the implication was not a clear object even for beginner teachers. Moreover, they showed that, contrary to a widespread idea, a logic lecture is not enough to get rid of these mistakes and difficulties.

Following these comments, we formulated the research hypothesis: it is necessary to establish links between these three points of view on the implication for a good apprehension and a correct use of it. In this paper, we make the assumption that a *didactic engineering* [Artigue M. 1990 & 2000] linking those three points of view can be built. In the following paragraphs, we present, therefore, some of our choices for this *didactic engineering* and some of our results.

CONDITIONS OF THE *DIDACTIC ENGINEERING*

The problem we present results from an experimentation carried out in 2001 with training teachers in mathematics. We worked with two groups of approximately 25 students at the IUFM [1] of Grenoble (France). This experimentation includes two three-hour-sessions on the proof and, in particular, on the implication. The first session contained two problems (one in geometry [Deloustal-Jorrand, 2002], one on pavings [Deloustal-Jorrand, 2004 a]), the second one proposed a work on written proofs [Deloustal-Jorrand, 2004 b]. We present, in this paper, the results of the analysis of the answers to the problem of geometry (first session). Before that, we display a mathematical and didactical analysis of the first question of this problem.

MATHEMATICAL STUDY OF THE PROBLEM OF GEOMETRY

In this paragraph we show that this problem of geometry, using only easy properties, may question the reasoning in a non obvious way.

Here is the first question of our problem of geometry[2]:

> Let ABCD be a quadrilateral with two opposite sides with the same length. What conditions must diagonals satisfy to have : two other parallel sides ?

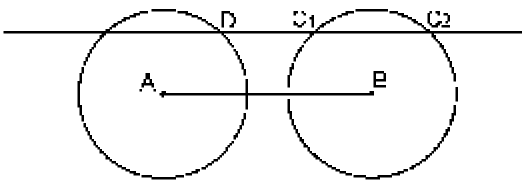
Let us call H the property “to have two opposite sides with the same length” and B the property “to have two other parallel sides”. We call *H* and *B* the sets which respectively represent them. The problem is now to find A with (H and A) $\Rightarrow$ B. Let us present two approaches which may induce different solving strategies.

The first approach raises the question of sufficient conditions. First example, one may list conditions on diagonals (same length, perpendicular…) and then check if these conditions, added with the hypothesis H, imply the conclusion B. This approach puts back the problem within the deductive point of view. Second example, one can also refer to known objects. Some quadrilaterals which satisfy both H and B are well known, for example squares, rectangles, parallelograms. Besides, the properties of their diagonals are also well known, and then one can work directly with equivalences. However, if some conditions may be cheaply found, these strategies do not give the exhaustiveness, all the conditions are not *a priori* reached.

The second approach raises the question of necessary conditions. Which objects satisfy both H and B ? Then, what properties A have their diagonals ? This approach seems natural and is basically related to sets point of view. Indeed, one must consider the set ***HÇB***. There are two ways to study those objects which satisfy H and B, either to be in ***H*** and add the property B, or to be in ***B*** and add the property H. Let us describe, in details, this first strategy, using a sets point of view.

Sets point of view strategy : H then B (H : two equal opposite sides)

Once the points A and B placed in the plane, the hypothesis (H), AD=BC, means that the points C and D belong to two same-rayed circles respectively, one centred on B, the other centred on A. Once D placed, the property (B) "two other sides parallel", means that C is the intersection of the straight line parallel with (AB) containing D with the centred on B circle. There are two intersection points C_1 and C_2.



Two configurations are thus obtained : isosceles trapezium (ABC_1D) and parallelogram (ABC_2D) [fig.1]. But one must not forget that, once A fixed, one may still change the distance AB, the ray of the circles and the position of D (linked to that of C) on its circle. So, when the two circles intersect, there is a new configuration : a cross quadrilateral called CQ_1 (ABC_1D) [fig.2]

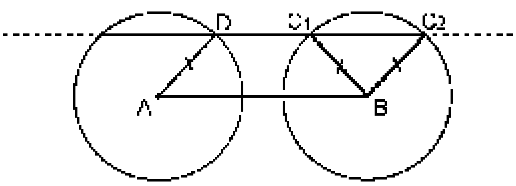


Fig.1

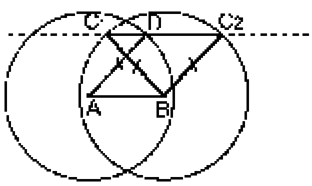
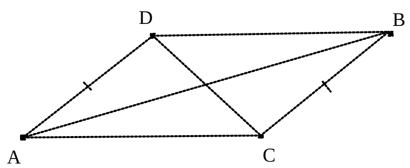


Fig.2

So there is the implication : (H and B) $\rightarrow$ (parallelogram or isosceles trapezium or cross quadrilateral CQ_1). We thus know the configurations which satisfy both H and B, it remains then to find the conditions on the diagonals.

However, for a quadrilateral, being a parallelogram is equivalent to having diagonals which cross in their middle. Isosceles trapeziums and cross quadrilaterals CQ_1 have same-lengthed diagonals. Now remains to see whether "to have same-lengthed diagonals" (A_1) is a sufficient condition, i.e. if the implication in the quadrilaterals: (H) and (A_1) $\rightarrow$ (isosceles trapezium or cross quadrilateral CQ_1) is true.

For that, the sets point of view is necessary again, we have to study the quadrilaterals which satisfy (H) and (A_1).We will not detail the rest of the solving, but let us say that these two conditions bring obviously the isosceles trapezium and the cross quadrilateral CQ_1 but also a cross quadrilateral CQ_2 (cf. here below) which does not satisfy the conclusion (B). The condition "to have same-lengthed diagonals" is thus not sufficient and will have to be restricted to exclude CQ_2. The final solving of this exercise is not the subject of this article, but we wanted to show how this problem can question the implication.



Cross parallelogramm CQ_2

DIDACTICAL STUDY

We present, first, some general choices for our *didactic engineering*, then the choices concerning, more precisely, this problem of geometry. Let us call again H the property "to have two opposite sides with the same length" and B the property "to have two other parallel sides", respectively *H* and *B* the corresponding sets.

Mathematical framework for our *didactic engineering*

First of all, we choose, for our experimentations, very easily accessible mathematical concepts. Indeed, our hypothesis is that, to see a work on the reasoning and distinguish difficulties due to the concept of implication, there must not be difficulties linked to a mathematical concept. This problem contains only notions very well known by students such as quadrilaterals, parallelograms, diagonals...

Real question

Our hypothesis is that the question must be difficult enough to allow a work on the reasoning. Besides, the truth of the implication must be questioned. Thus, we ban questions like: "Prove that A $\Rightarrow$ B is true".

Implication versus equivalence

A problem whith equivalences (⇔) does not allow a work on the implication. We choose for our experimentation to distinguish a necessary and a sufficient condition.

Practical organization of the sessions

Our hypothesis is that a research in groups is necessary for our *didactic engineering*. That allows a confrontation between the various points of view. Nevertheless, the first individual work gives each one the time to make his own opinion about the problems. These various opinions will feed the discussions.

Choices to put forward the three points of view in our *didactic engineering*

We made the assumption that the "deductive reasoning" is usually always present. Therefore, in our *didactic engineering*, we chose to emphasize the sets point of view and the logical one, depending on the problems.

Choices to put forward the sets point of view in this geometry problem

The objects questioned in this problem are not often considered in the french secondary school.

First, *H* is the set of the quadrilaterals having two equal opposite sides. Their properties are not as well known as those of parallelogramms for example. Yet, in french secondary school, one usually considers, in fact, the implication "in *H*, A $\rightarrow$ B". Here *H* is implicit because very well known and used. For example, most of the time in the parallelograms' class, properties are implicitly used (for example, convexity). Here, the property H must be explicit during all the resolution.

Besides, *H*, *A* and *B* contain crossed quadrilaterals which are not teached in french secondary school. The training teachers must distinguish the crossed quadrilaterals and define them.

Therefore, the presence of these quadrilaterals should push the strategies linked to the sets point of view. Indeed, the training teachers' usual knowledge on the quadrilaterals is not sufficient to give a correct answer.

Choices for a work on the implication in this geometry problem

The condition A is unknown. Hence, the deductive reasoning is "upended" here.

It is hard to find A by using the reasoning: "in H, $\neg$B $\Rightarrow$ $\neg$A". Indeed, it is difficult to define the set corresponding to $\neg$B "Not to have two other parallel sides".

The condition A sought is, in fact, equivalent to (A_1 or A_2). Usually, the taught implications are like (H **and** A1 **and** A2) $\Rightarrow$ B, whereas, here, it is (H and [A1 **or** A2]) $\Rightarrow$ B. Consequently, we think A is more difficult to define and prove.

At last, the problem itself forces to question the direction of the implication ($\Rightarrow$, $\Leftarrow$). Indeed, we did not specify if the requested conditions were necessary or sufficient.

Paper Vs. interactive geometry software in this geometry problem

A part of the task consists in varying the parameters of the figure (length of the ray, position of D on the circle…). An interactive geometry software (cabri for example) would make this part of the resolution easier. However, we assume that the use of this software would prevent us to check whether the sets point of view is used or not. Indeed, we would not be able to see if the training teachers are working on the sketch itself (as seen on the screen) or on the figure (as a representation of the sets). In fact, with the software, the manipulation of the sketch is mainly due to the "didactical contract". That is to say, the user has no choice but to manipulate the sketch. Consequently, this manipulation is not, most of the time, the result of a "mathematical choice" and is not a clue of the presence of the sets point of view. Hence, we chose to work on paper rather than with a software.

RESULTS

Working on the implication

First of all, we can say there was a real mathematical question in this problem. Although students first thought this exercise very easy, its solution required a very long research in groups. The answers are incomplete and, in the end, the students declared this exercise very difficult.

Robert: It is an exercise which as a teacher, I would not give before university

On one hand, no group found the problem obvious but, on the other hand, no group was stopped by mathematical difficulties. They had strategies to begin their research. That is why we can assert that a work on reasoning and implication was done.

The exercise fulfilled its role, as for the work on the implication. Indeed discussions about necessary and sufficient conditions took place in the groups during all the research. The teachers looked, first, for sufficient conditions but, because of the problem, they had to look for necessary conditions too.

Anne: We looked for sufficient but…if we consider it, maybe…it is necessary too.

Thus, there were a lot of questions in all the groups. For example, they all wondered if the property "to have same-lengthed diagonals" was a sufficient condition. We saw then teachers confusing sufficient condition and necessary condition.

Antoine: No, « same-lengthed diagonals » is not sufficient. A parallelogramm is OK, yet its diagonals don't have the same length.

This counter-example is not valid, it shows that the condition is not necessary but it does not show anything on the sufficient quality.

Besides, all the groups were looking for a "complete" condition merging all the possible conditions. They all found the sufficient condition "to have diagonals which cross in the middle" but they all thought it was not enough to answer the problem.

Armelle: I was looking for a condition that would meet…all possibilities.

Moreover, most of the groups tried to define a “minimal” condition, which requires the fewest hypotheses. Then, they had to take the property H into account.

Anne: Until now, we didn’t use AB=CD ! Thus if we take into account this new condition, maybe we can formulate more easily our condition on the diagonals.

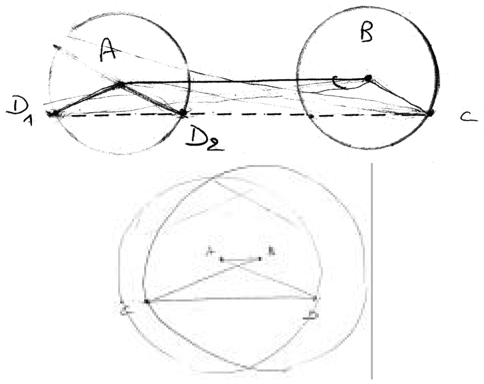
Furthermore, as we had planned, they were not able to use the reasoning “in H, ¬B ⇒ ¬A” in order to find A. But one group tried and had to argue about it.

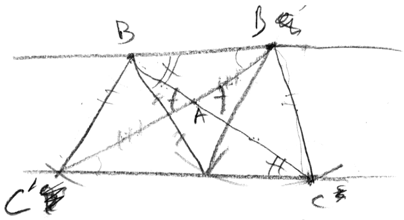
Davy: But in fact…no, the problem is, what are you going to look for on your diagonals ? What are you going to deny on your diagonals ?

Working on the sets point of view

Most of the groups did not know *how* to solve this problem. This confirms our presumption that the ensemblist point of view is necessary in this situation. The analyse of the answers shows that the sets point of view is not an available tool for the training teachers. But on the other hand, we can see a lot of implicit clues of its presence. We now want to describe these clues.

First of all, all the groups used sketches in their strategies. Some of them are similar to those of the strategy based on sets point of view in the mathematical analysis.





Nevertheless, they did not check the variation of their sketches (length of AB, BC…). As a result, they did not check that their sketches represented all the possible quadrilaterals. Thus, they can not be sure that they found all the possibilities for ABCD. Yet, these sketches show that the quadrilaterals were built as sets of points.

Besides, the conjectures, examples, counter-examples are very present. These are marks of the sets point of view since teachers have to speak about sets.

Furthermore, the teachers often forgot the property H “two same-lengthed sides”, especially when they seeked counter-examples.

Laura : But, your opposite sides with the same length, where are they here ?

Antoine : Oh yes, I made a mistake.

Yet, before long, the situation itself and the work in groups allowed them to see that the property H was not taken into account. No group kept a false counter-example. This property fulfilled its role since the teachers had to remind it during the resolution and had to consider the set of quadrilaterals which satisfy H.

Despite it, the sets point of view is not a tool for the teachers as we show now.

All the groups left apart the crossed quadrilaterals, explicitly or implicitly when they began their resolution. Some groups took them into account afterwards, but they separated this new research. They gave some reasons: crossed quadrilaterals are not interesting, they are particular objects, they do not have diagonals…

Robert: But why do you draw a crossed one, it is not an interesting case!

Carine: You can't talk about diagonals if you link this with this. Thus necessarily it is not crossed.

Besides, the sketches are not seen as a tool of the set point of view. Then they can not be considered as a proof. Yet, all the teachers did use sketches. That is why, one of the main questions during the resolution, in all the groups, was to find what could be the role of these sketches. Do they give all the possible quadrilaterals ? If there are enough sketches can one be sure to have all the quadrilaterals ? Can they be considered as a proof ?

Laura: To prove is not to draw !

For the training teachers, a sketch can not be a proof whereas this can be true in the sets point of view as we showed in the mathematical analysis.

Working on the logical point of view

Most of the groups gave different sufficient conditions. Only one group gave a single condition written with the logical word "**or**". Generally, the equivalence $[A_1 \Rightarrow P \text{ and } A_2 \Rightarrow P] \Leftrightarrow [(A_1 \text{ or } A_2) \Rightarrow P]$ is not admitted, even when proposed by a teacher. This situation allows a discussion on this equivalence.

CONCLUSION

We have shown that this problem allows to question the implication and the proof. Indeed, the training teachers have difficulties which are not related to mathematical objects. They had to examine the conditions to know wether they were sufficient or necessary. They discussed about what a mathematical proof is.

Moreover this problem requires a work under the sets point of view. Although it is not used as a tool, it appears many times, concerning counter-examples or the role of sketches. Besides, the logical point of view appears too, especially to express the final condition with the "or", but also to find a minimal condition or sometimes to decide if a condition is sufficient or necessary.

Lastly, these results are to be placed among others. Indeed, this problem of geometry forms part of a six hour experimentation including other stages of work, in particular, studies, in groups, of written proofs [Deloustal-Jorrand, 2004 b] and of a problem of discrete mathematics [Deloustal-Jorrand, 2004 a]. Moreover, this experimentation takes sense when one knows that it was preceded by two others, carried out in 1999 and 2000.This problem of geometry is, thus, to consider as part of a broader context.

NOTES

1. University Institute for Teacher Training

2. There were two other questions: two 90 degrees angle ? ; two other same-lengthed sides ?.

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