REVIEWING TEXTBOOK PROOFS IN CLASS: A STRUGGLE BETWEEN PROOF STRUCTURE, COMPONENTS AND DETAILS

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Abstract. During the first year of a university study in mathematics, "dissection" of mathematical proofs occupy a growing part of the course time. In this paper I investigate how we can describe, characterise, analyse and thus understand what is going on during a presentation of a textbook proof in class. The conclusion is that students' misunderstandings and miscommunications between teacher and students may be explained if the analysis separates between the proof structure, the components in the structure and the details of the proof. Excerpts from a presentation of a proof in an analysis course at a Danish university are used to illustrate this point.

INTRODUCTION

A traditionally taught university course is often divided between lectures where the content of the textbook is explained by the lecturer and exercise/problem solving sessions where the lecturer or a teaching assistant works through assigned tasks. Gradually during a university study in mathematics, "dissection" of proofs (investigating and analysing how different parts of the proof function and how they relate) becomes a more and more important activity in the teaching practice[1] and it is for that reason interesting to investigate this activity in detail. In this paper I want to focus on the following question:

- How can we describe and characterise a teacher's presentation of a proof in the classroom and a dialogue with the students?

The work presented in this paper is part of a larger exploratory project concerning mathematics teaching and problem solving at the tertiary level baring the research question: *In what ways does the teaching practice influence the way students approach and solve mathematical tasks?*

The mathematical subject is moderately advanced mathematical analysis (beyond calculus) where the mathematical tasks demand justification and/or a proof of a claim about properties of or relationships between mathematical objects and concepts. At this mathematical level proof dissection is a very important part of the teaching practice and I want to use this paper to illustrate how a proposed theoretical tool can be used to describe, analyse and characterise textbook proofs and the part of the teaching practice that involves proof reviews.

Although it is common knowledge in the mathematical community that understanding a mathematical proof takes more than just verifying each step in the proof (Bourbaki, 1950), classroom practice is more concerned with analysing the details than with synthesizing, "combining parts to a whole" (Dreyfus, 1991). The discussion of how to present mathematical proofs to students is (naturally) still an ongoing topic in educational research concerning proofs, both with respect to analysing and categorizing "normal" teaching styles (Weber, 2004; Hemmi, 2006) and to suggesting alternative ways to teach proofs (Leron, 1983; Balacheff, 1991; Alibert and Thomas, 1991; Legrand, 2001) often emphasising the "conviction part" of the purpose of proving (Harel and Sowder, 1998). The reason for such research is that "*… the key role of proof is the promotion of mathematical understanding, and thus our most important challenge is to find more effective ways of using proof for this purpose.*" (Hanna, 2000, p. 5-6).

Beside research concerning the teaching of proofs, many research studies concern documentation and analysis of students' difficulties with constructing mathematical proofs (Moore, 1994; Dreyfus, 1999; Weber, 2001; Selden and Selden, 2003) or their perceptions of what constitutes a valid proof (Martin and Harel, 1989; Dreyfus, 1999; Healy and Hoyles, 2000; Raman, 2003).

The literature does not however offer a comprehensive framework for analysing and comparing teaching practices (social perspective[2]) in relation to students' proof production processes (individual perspective[2]). I found it necessary and useful to develop a framework that could be used to analyse my data material. The construction I propose is based on data (a "bottom-up" approach) and mathematically grounded, and can, beside being used in the analysis of classroom proof presentations, also provide a tool for analysing students' proof production processes, thus allowing for a way to relate the teaching of proofs to students' proof construction difficulties (the latter feature, though, is not demonstrated in this paper).

THEORETICAL CONSTRUCTION

The important notions in the proposed theoretical construction are the notions of *structure*, *components* and *details*. A structure is composed of interrelated components. The specific details of each component can vary in number and complexity. When talking about a textbook proof the following definition of the structure, components and details is suggested:

The structure of a proof is a hierarchical network consisting of the main steps or components in the chosen proof strategy. The elements of the realisation of the components are called the details of the proof.

In a situation where a student has to construct a proof by herself, she has to decide on a proof strategy, construct the proof in a number of sub-steps and finally provide the details of those steps. When the proof is already made as the case is in a textbook the student has to identify the proof strategy used, the components the proof are made up of and the details of these components. In the proposed definition, the structure of a proof equals the hierarchy composed of the strategy choice, the components and the details. The main steps in a proof are often related in some way, but the details of one component may, besides having a relation to other details in the same component, also relate to details of other components in the structure. Relations between components and relations between details of different components give rise to a network within the hierarchy.

There is a dialectical relationship between structure, components and details. It is not possible to comprehend the proof structure if the components are not known and to identify something as a component implies that it is a component of a larger system. Similar considerations apply to the details of the structure.

Although this construction bears some resemblance with "the structural method" proposed by Leron (1983), where a proof is regarded as composed of levels which again consist of modules containing "one major idea of the proof", it is however different. In the structural method the first level contains the big lines in the proof without any technical details, whereas the last level contains all the specifics in the proof. The components in a proof, as defined in the proposed theoretical construction, can therefore not be equated with the levels in the structural method. And more importantly, the structural method is not a tool for analysing a proof presentation that is based on the "linear method", as defined in (Leron, 1983).

With the suggested tool it is possible to analyse if the teacher and the students talk about the structure, the components or the details during the presentation and discussion of a proof in the classroom. Within this theoretical framework it is possible to pose a hypothesis for the larger project: *Confusion about what is structure, components and details in the teacher's dissection of a proof can account for students' difficulties solving tasks.* In this paper I first consider a concrete textbook proof with the intention to identify structure, components and details and then I present the analysis of the presentation of the proof in class.

ANALYSIS AND CHARACTERISATION OF A TEXTBOOK PROOF

The analysis course I observed used the textbook "An Introduction to Analysis" (Wade, 2004), so I stick to the formulation of the proof from this book. The exact wording of the theorem is:

> **Theorem 3.6** [Sequential Characterisation of Limits]
>
> Let $a \in \mathbf{R}$, let I be an open interval that contains a, and let f be a real function defined everywhere on I except possibly at a. Then
>
> $$L = \lim_{x \to a} f(x)$$
>
> exists if and only if $f(x_n) \to L$ as $n \to \infty$ for every sequence $x_n \in I \setminus \{a\}$ that converges to a as $n \to \infty$. (Wade, 2004, p. 60)

The proof of theorem 3.6 is given below. The proof refers twice to an implication (1) from the definition of limits of functions, definition 3.1:

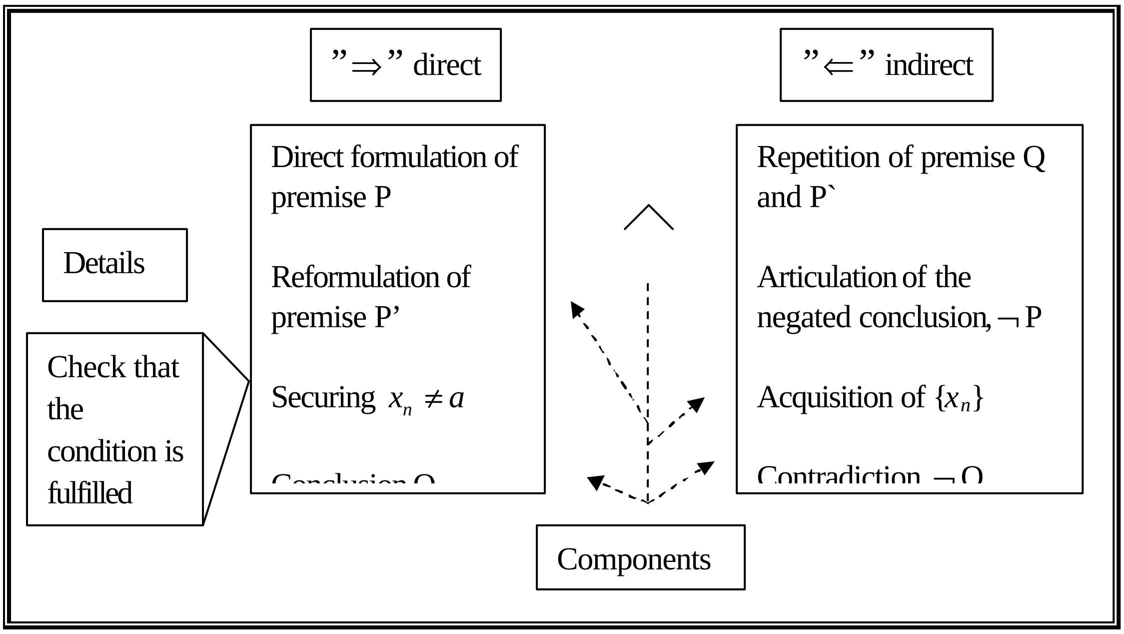
> **Definition 3.1**
>
> Let $a \in \mathbf{R}$, let I be an open interval that contains a, and let f be a real function defined everywhere on I except possibly at a. Then $f(x)$ is said to *converge to L*, as *x approaches a*, if and only if for every $e > 0$ there is a $d > 0$ (which in general depends on e, f, I and a) such that
>
> (1) $0 < | x - a | < d$ implies $| f(x) - L | < e$.
>
> (Wade, 2004, p. 58)

I have included numbers in the proof as a help for the analysis, but besides those numbers the proof is a verbatim reproduction of the textbook proof:

> **Proof**
>
> **1)** Suppose that f converges to L as x approaches a. Then given $e > 0$ there is a $d > 0$ such that (1) holds. **2)** If $x_n \in I \setminus\{a\}$ converges to a as $n \to \infty$, then choose an $N \in \mathrm{N}$ such that $n > N$ implies $| x_n - a | < d$. **3)** Since $x_n \neq a$, **4)** it follows from (1) that $|f(x_n) - L| < e$ for all $n > N$. Therefore, $f(x_n) \to L$ as $n \to \infty$.
>
> **5)** Conversely, suppose that $f(x_n) \to L$ as $n \to \infty$ for every sequence $x_n \in I \setminus\{a\}$ that converges to a. **6)** If f does not converge to L as x approaches a, then there is an $e > 0$ (call it e_0) such that the implication "$0 < |x-a| < d$ implies $| f(x) - L | < e_0$" does not hold for any $d > 0$. **7)** Thus, for each $d=1/n$, $n \in \mathrm{N}$ there is a point $x_n \in I$ that satisfies two conditions: $0 < | x_n - a| < 1/n$ and $| f(x_n) - L | \geq e_0$. **8)** Now the first condition and the Squeeze Theorem (Theorem 2.9) imply that $x_n \neq a$ and $x_n \to a$, so by hypothesis, $f(x_n) \to L$, as $n \to \infty$. In particular, $| f(x_n) - L | < e_0$ for n large, which contradicts the second condition. (Wade, 2004, p. 60)

Theorem 3.6 includes a bi-implication ("if and only if") and the majority of proofs of such theorems are structured in two parts where one implication is showed at a time. The chosen strategy is to prove the first implication "$\Rightarrow$" with a direct proof whereas the second implication "$\Leftarrow$" is proved indirectly by contradiction. To make a strategy choice or to understand why a given strategy choice has been made is an important part of a strategy discussion. In the textbook this strategy choice is not emphasised or discussed.

The theorem has a twist because there is a double hypothesis part. There are thus *two* premises in the first part; P: "$f(x) \to L$ *as* $x \to a$" and P`: "$x_n \to a$ *as* $n \to \infty$" and a conclusion Q: "$f(x_n) \to L$ *for* $n \to \infty$". The proof strategy of the first implication is thus: "if P and P`, then Q", i.e. (P $\wedge$ P`)$\Rightarrow$ Q. In the first step the premise P is directly formulated, while premise P` is reformulated in the second step. One might at first glance think that the two steps are similar, but the second step deviates from a mere formulation of the premise. It draws the consequences of premise P` and is in that sense a reformulation of P`. The third step provides the missing link before the results so far can be combined; namely securing that $x_n \neq a$. In the fourth step, the combination of the formulation of premise P, the reformulation of premise P` and the securing leads to the conclusion that $f(x_n)$ converges to L. The structure of the proof with the described components is shown in figure 1.



Figur 1. The structure of the proof is composed of the main steps or components that the chosen proof strategy leads to. The realisations of the components are the details of the proof. The details are shown for one of the components as an illustration. The notation used is: P: $f(x) \to L$ as $x \to a$; P`: $x_n \to a$ as $n \to \infty$ and Q: $f(x_n) \to L$ as $n \to \infty$.

In the second part of the proof an indirect proof strategy, proof by contradiction, is chosen for non-explicit reasons. P` is still a premise, but now Q is a premise and P is the conclusion. The logical structure of this part is based on the logical tautology $[(Q \wedge P` \wedge \neg P) \Rightarrow \neg Q] \Rightarrow (Q \Rightarrow P)$. Since the data excerpts only concern the first part of the proof I will not go further into the analysis of the second part.

What does it take to realise the different components? What are the details? I will give some examples. In the first component the formulation of premise P demands a reproduction of the definition of the limit of a function, which includes a repetition of the definition and a switch between the different formulations, phrases and notations used to describe limits of converging functions. The details of the component where we make sure that $x_n \neq a$ (the third component) is just a contemplation that this condition is fulfilled.

It is not uniquely determined what should be the content of the structure, components and details in a proof. We shall see, however, that the proposed characterisation can help characterise what is going on in the classroom.

ANALYSIS AND CHARACTERISATION OF THE CLASSROOM PRESENTATION

Data for the project was constructed through non-participant observations (Bryman, 2001) of a four month long traditionally taught real analysis course at a Danish university. I have selected two sequences from the 25 minutes long presentation of the above proof. Between the excerpts I give a summary of what takes place in the classroom in the non-documented periods so the reader will get a sense of the whole proof presentation. The students were expected to have read or browsed through the proof before the lecture and they were not going to have a test in the proof.

The teacher begins the proof with a claim that proving the first implication is almost trivial (30-33). He says that since they have to talk about all sequences they need to pick an arbitrary converging sequence and see what they can say about that one (34-38). Then he proceeds to make a graphical illustration of the situation (39-40). We enter the scene where he comments on his illustration (the teacher uses a different notation than the textbook, a instead of L and x_0 instead of a). In the excerpt the teacher hastily goes through the first two components (41-42). Then he jumps to the component of the conclusion Q (43) and finally back to the details of the second component (44-46):

41	Teacher:	We have a graph f. We have an e window. We have a d which matches. … and we have a sequence, eh, x_n converging down to x_0 and we want to show that the function values of the sequence converge to a, right? And what does it mean that the sequence converges to x_0? … well, then it has to stick to this interval, minus d to d, as long as n is big enough. Mary, isn't it?
47	Mary:	I was just gone there for a moment ..
48	Teacher:	You were just, yes, okay. We want to show that the sequence of function values f of x_n converges to a and what we know is that if x is in the d interval around x_0, then all the function values are in the e interval around a. And then I say, if we are to make sure that $f(x_n)$ is at most e away from a then it is basically enough to capture x_n in this interval from minus d to d because then we know that the function values are in the right interval … and there .. Can we make sure that x_n is in the interval from x_0-d to x_0+d?
56	Susan	Has it something to do with choosing an n that is big enough?
57	Teacher	That sounds like a really good idea. Can we do that?
58	Susan	We can do that.
59	Teacher	We can do that. What, eh, how big does it have to be?
60	Tom	Bigger than capital N.

61	Susan	Yes, it has to be bigger than capital *N*.
62	Teacher	No, it's capital *N* that we are about to choose, right? How big are we going to choose capital *N*?
64	Paul	So big, that the difference between the sequence and the limit is less than, numerical, less than *d*.
66	Teacher	Than *d* …

After Mary's sign of lack of attention (47) what is the teacher then doing? He begins "backwards", starting with conclusion Q (48-49) which is followed by the first component, "formulation of premise P" (49-51). Then he tacitly reformulates the logical structure of the proof (51-53): "*if* Q needs to be true, *then* it is enough if P` is true". Instead of talking about the necessary condition for Q to be true ("if P and P`, then Q"), he now focuses on a sufficient condition and that draws attention to premise P` instead of conclusion Q. It is (presumably) very difficult for a student to follow this equivalent reformulation when the teacher does not explicate what he is doing.

The teacher involves the students on five occasions (in 45-46, 54-55, 57, 59 and 62-63). On two of those occasions (54-55 and 57) he poses a question where a proper answer would refer to the second component, "reformulation of premise P`": "*yes, because* $\{x_n\}$ *is chosen to be a converging sequence*". The first reply from Susan refers in stead to the details of this component and in her second reply she does not justify her answer. On the three other occasions the teacher asks with reference to the details of the second component and this is also the response he gets from the students.

This way of analysing the excerpt shows that the teacher aside from tacitly reformulating the logical structure of the proof also shifts between a component perspective and a detail perspective. The students maintain a focus on the details.

The teacher writes down the details of the first two steps (66-72). The following excerpt concerns the securing component. The details of this step only include an inspection which explains why the teacher characterises this step as "free" (75):

73	Teacher	… And then I quickly just want to add, that zero is less than the distance from x_n to x_0 and that is because my sequence will never reach the value x_0, right? That is just for free.
76	Susan	That is just for free?
77	Teacher	Yes, that is, it's just there, my sequence was contained in I without x_0, so none of the x_n's can be x_0.
79	Susan	Why is that free?
80	Teacher	Well, I mean, that assures me that the distance is bigger than zero. That's what's free. When I have paid the other price first, right?

Supposedly, Susan does not realise the details of this component because the structure of the proof is not clear to her and she does not recognise what role the component plays in the structure. Her uncertainty about the structure and to which part of the structure the discussion is located makes it impossible for her to comprehend the details of this component.

The teacher finishes the first part of the proof (82-86) and they have a discussion about the notation (87-114). The teacher moves on to the second part of the proof where he proclaims that he wants to make it as a proof by contradiction if none of the students have any other suggestions (126-128). So in his presentation of the proof no emphasis is put on the justification of the strategy choice in neither the first nor the second part of the proof. After repeating premise Q and conclusion P (129-132) the teacher guides the students through the details of the component "articulation of the negated conclusion" (133-165). Here both the teacher and the participating students are talking about and referring to the details and it is clear from the transcripts (not shown) that the students are able to follow his guiding. A reason for this accordance may be that the students recognise the link between the strategy choice and the negation component and thus are able to understand the explanation of the details.

After guiding the students through the details of the negation component the teacher continues to the seventh step, the "acquisition component", which leads to difficulties for the students. He begins with a repetition of premise P` and Q (166-167). A student expresses difficulties with the choice of the sequence *1/n* and the teacher tries to explain it while maintaining a focus on the details (184-209). After trying to explain the acquisition component the teacher interprets a question from a student as a formulation of the contradiction component (210-215) and the teacher quickly summarises the components of the second part of the proof (215-220) and writes down a formal version of his summary focusing on the details (216-237). Then Susan expresses some confusion about the structure of the proof; didn't they assume what they were trying to prove? This leads to a clarification of the logical structure of the second part of the proof (238-247) and of the logical structure of a proof of an arbitrary "if-then" theorem (248-254).

SUMMARY

The two chosen excerpts show episodes where the teacher and the students in some way miscommunicate, but I briefly mentioned one example where the students and the teacher had a united approach, namely in the formulation of the negated conclusion. In the first excerpt the teacher jumps around between the components and he reformulates the logical structure of the first part of the proof. To a student who does neither comprehend the structure of the proof nor is able to separate the components from each other it must be nearly impossible to follow the presentation and to comprehend the details. In the second excerpt the teacher explained the details with reference to the (underlying) structure. In order to understand why the fulfilment of the relation is "free" it is necessary for the student to see what role this component plays in the proof. In the last part of the presentation two students expressed difficulties understanding the details of the seventh component even though the teacher tried to explain the details. Utterances in the excerpt and later in the presentation indicated that the proof structure was not clear to at least one of the students, so again it is possible to conclude that a lack of understanding of the proof structure and the components prevents a comprehension of the details.

As mentioned, the notions in the framework are dialectical related. As the analysis of the transcripts shows, this dialectical relationship is in fact visible in the students' struggles to comprehend the proof.

Research studies show that university students typically exhibit difficulties handling quantification (Dubinsky, 2000). These difficulties are not directly addressed in this framework. The framework has been constructed through a "bottom-up" process founded on data and it is thus context dependent. Difficulties with quantification did not appear to be as essential in explaining the students' difficulties as their struggle to separate between structure and details in the dissection of a proof in class.

NOTES

1. By teaching practice I am referring to activities taking place in the course session time, to the organisation of the course, to the choice of textbook, to the way the subject matter is presented and to communication in class.

2. The framework developed by Cobb and co-workers takes both the social and the individual perspective into account (Cobb et. al, 1997). I did not find this framework completely useful for my data analysis because of a lack of focus on solving processes. I however found use of this framework for parts of the data analysis (not reported here).

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