MATHEMATICAL PROOF:
TEACHERS' BELIEFS AND PRACTICES

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The gap between the mathematical curriculum and what is actually taught in classrooms is an educational worry that requires closer investigation. Teachers' beliefs can possibly throw some light on the reasons explaining this gap. This paper discusses results of my masters' research, illustrating how teachers' beliefs play out in their practices and focuses on the ways these conceptions influence, particularly, the teaching of mathematical proof. The paper aims to point out two teachers' different views of what constitutes proof and the functions of proof they chose to integrate into their teaching practices. Finally, this research sketches some educational implications to improve teachers' - and consequently students' – performances in relation to proof in mathematics.

INTRODUCTION

Teachers' beliefs play a fundamental role in effective mathematics teaching. Most researchers in the area have examined primary school or pre-service teachers' beliefs and practices (Foss & Kleinsasser, 1996; Thompson, 1992; Ernest, 1988; Hanna, 1989). Yet, few examples can be found in the literature about specific subject matter knowledge and beliefs (Ball, 1990; Even, 1993; Tirosh & Graeber, 1990) and fewer about beliefs and proof (Jones, 1997; Hoyles, & Küchemann, 2002). This study explores the relationship between beliefs and proof in the context of secondary school mathematics.

The general motivation for this study derives from my need to call into question the idea that "teachers teach the way they have been taught" (Frank, 1990, p. 12). Mathematical research (Pepin, 1999; Knowles, 1992, Borko, Flory & Cumbo, 1993) has shown that teachers' beliefs are formed during their schooling years, are shaped by their experiences as pupils and hardly change. Furthermore, teachers' conceptions and feelings are revealed during their lessons and affect their decision-making (Woods, 1996), goals (Nespor, 1987), task-defining (Pajares, 1992), priorities (Aguirre & Speer, 2000) and their overall pedagogical approach. As a result, it is questionable whether all students are taught the same mathematics. The students' knowledge and skills are dependent on teachers' beliefs of the mathematical content to be taught. The challenge for the educational community is to provide appropriate training to teachers in order to help them reflect and control the influence of their personal conceptions of mathematics on their practices.

The paper is divided into three sections: in *Section 1* the theoretical framework is discussed; in *Section 2* research tools and methodology are explained; in *Section 3* an analysis of results is presented and, finally, some conclusions are sketched at the end.

1. THEORETICAL BACKGROUND

Teachers' beliefs

Beliefs are defined as conceptions, personal ideologies, world views and values that shape practice and orient knowledge (Ernest, 1989; Thompson, 1992). Teachers often resist adopting educational changes because "changing beliefs causes feelings of discomfort, disbelief, distrust and frustration" (Anderson & Piazza, 1996, p. 53). Nevertheless, recent researchers (Kagan, 1992; Franke *et al*., 1998) argue that lasting changes may occur if teachers try new strategies in their classrooms and reflect on their own belief systems. However, according to Richardson (1996, p. 114) "it cannot be assumed that all changes in beliefs translate into changes in practices".

The relationship between teachers' beliefs and practices is complex. Pepin (1999) found that teachers' conceptions of mathematics and its teaching and learning are not related in a simple cause-and-effect way to their instructional practices. Foss & Kleinsasser (1996) described this relationship as symbiotic; Cohen (1990) identified inconsistencies between teachers' professed beliefs and teaching. The key issue is to find ways to increase teachers' awareness of their beliefs, conceptions and ideas about mathematics. This paper focuses on a specific mathematical aspect, the proving process.

Mathematical proof

Proof can be defined as "ways of convincing someone else of the truth of a statement" (Gutierrez & Jaime, 1994, p. 3). Students often have poor performance and understanding in mathematical proof. According to Schoenfeld (1994, p. 75), "in most instructional contexts proof has no personal meaning or explanatory power for students". Also "students judge that after giving some examples which verify a conjecture they have proved it" (Hoyles, 1997, p. 7). Many of the students' difficulties are due to confusions resulting from their teacher's approaches to proof. Ernest (1988), among others (Thompson, 1984; Calderhead, 1996; Cohen, 1990), claims that teachers' performance is highly depended on their system of beliefs. Therefore, it is vital to examine what kind of conceptions of mathematical proof and knowledge teachers hold because, as Jones (1997, p. 16) states, "the successful teaching of mathematical proof depends crucially on the subject knowledge of mathematics teachers".

2. RESEARCH TOOLS AND METHODOLOGY

This study was carried out in Bristol, UK. Two secondary teachers – George and Nicky – selected purposely, were observed carrying out two lessons each and were interviewed based on pre-observational tasks (concept map and proving task[1]). The

[1] Retrieved from the "Longitudinal Study of Mathematical Reasoning" (1999-2003) project (Year 8 activities) and modified.

tasks encouraged them to talk about their ideas, understanding and conceptions about the *nature* and the *function* of proof in the school context. Lesson observations showed the ways in which those beliefs were carried out, in respect to the tasks set and the questions asked by the teachers.

More specifically, the teachers firstly drew a concept map each to show their understanding of the nature of proof. Also, they completed a proving test, solving problems and providing marks for different sets of responses. This activity provided information about their teaching approach. Afterwards, I interviewed them based on those tasks to reveal their personal constructs of proof. The second phase included two observations of lessons about probabilities with Year 8 students for each teacher. The four lessons included activities with coins and dice and the possible outcomes, for example the number of heads and tails with 2 or 3 coins; the number of 5's and 6's with 2 or 3 dice etc. The observations gave a sample of teachers' instructional approach and behaviour in the classroom. Finally, the teachers reflected, commenting on those lessons. This process provided deeper insight into their beliefs' systems.

The research questions of the study were: (1) What are teachers' conceptions about the nature and role of proof in the context of secondary school mathematics? (2) What is the relationship between teachers' conceptions of proof and their practices?

In this paper I will try to explore only some aspects of these questions and show their relevance for the teaching of mathematical proof.

3. FINDINGS AND ANALYSIS

To answer the research questions I designed a theoretical framework which identified teachers' beliefs according to their responses. Particularly, the analysis is based on the different functions of proof: verification (Bell, 1976), explanation (Hersh, 1993), communication (Raman, 2003), discovery (Schoenfeld, 1986) and systematization (Knuth, 2002). Furthermore, the data analysis provided five clusters which allowed comparisons between the two case studies: a) beliefs about the nature of proof; b) beliefs about the functions of proof; c) discussions and group work; d) formal and semi-formal teaching approach to proof and e) classroom culture.

3.1. Case study #1: George

George is the Head of Mathematics. He has ten years of experience and he has got a Master's Degree in Mathematics Education. George is currently working on a PhD proposal. I observed him teaching two lessons with Year 8 students about probabilities. These lessons provided a snapshot of his practices in respect to proof. In this section I summarize his approach and beliefs about proof derived from the concept map, the proving task and the lessons.

Concept map

George used 14 key words in total to draw his concept map (see diagram 1 below). He started from the word "awareness" which he considers to be essential for proof. He links this key word with 9 other words related to proof – this is more than any other word used in the map. He continued with "open-ness" and then "beauty" and linked those two to "awareness". At that point, he added insight which also linked to "beauty" and "awareness". Then the other words followed. The word "proof" is in the center and there are not any links to it at all. Four out of thirteen key words are in *–ing* format (convincing, testing, experimenting and "trying to prove it wrong") and which he relates to classroom activities and context. George's vocabulary during the discussion about his concept map included other phrases or terms such as : incredible pleasure, I always encourage people, motivation to try to prove all cases, community of mathematicians, less rigorous, convince the community.

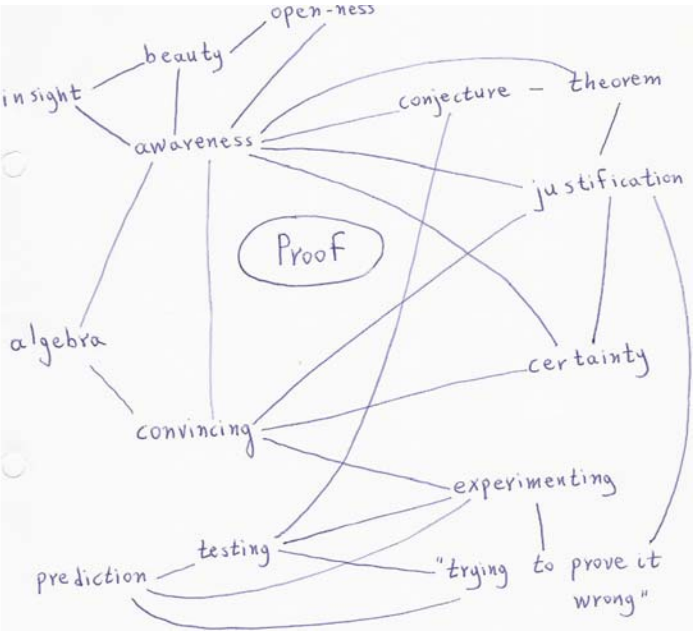

Diagram 1: George's concept map

Proving task

George finds Ben's answer the best one because it is the only one which provides awareness and explains "why":

> "I suppose it seems to capture the essence of "why", it seems that he got the key awareness of why he got 27 and he seem to describe that awareness very clearly. I didn't have an awareness why it was 27 when I was reading it and when I came to that I said: Oh yes!OK. So it promotes awareness in me about this problem which I suppose to me is what the best groups do."

He gave Ben the mark 10 out of 10 because "he had justified in terms of mathematical structure" and that is what the National Curriculum sets in the marking criteria. George admits that Ben's statement is not an axiomatic proof or absolutely rigorous but it is still a convincing proof in terms of communication. Amina got 4/10 because "she does not seem to have awareness of the problem and probably she has only convinced herself but not the others". Carol (1/10) and Davor (2/10) "are in a lower level of understanding the problem".

Lessons

Commenting on the lessons he had, George feels that he had useful conversations and that probability games helped students to change and learn.

> "Proof for me is not a separate mathematical activity, so I don't think I ever set out to teach proof as such […] if there is something interesting going on in my lesson then proof will be around I'm sure. so even for example doing probability it felt as so what we were discussing with the two coins was an aspect of proof: How can we be sure that the analysis into 1/4 , ¼ and ½ is the correct one? so to me proof is about convincing myself convincing others and as a class coming to some agreement about what we think is the case…and that really for me is what proof is […]"

He believes algebra is essential and important to proof because it answers many "whys" and this is the kind of classroom environment he tries to create. His instruction promotes understanding and the answering of students' questions, "why?"s. This was clear when I observed the language and the approach he uses in his lessons (*italics added*):

> "What do you mean by..? We need to assign in theory-theoretical probability-how these are likely to happen? Can you say *why*? Can anybody help us sort out the 26/52?"

> "Can anybody *apply* Jamie's idea in two dice? Is the Tail-Head same with Head-Tail? What actually happened when you played the game?[…] There is a *need* now (*at he end of the discussion*) to hear what people think..whether we need to have H-T or T-H probabilities"

> "Write a *prediction* about which column will win […]"

> "Your *challenge* is what is the probability for the other games…I suggest to start with 3 coins…What do you *expect* to see in these 4 columns …could you make some *predictions* with 4 dice and 5 dice?..Tom noticed that as we go from 2 dice to 3 dice we double the possibilities. Maybe we could find some *patterns* […]"

George explains that alongside proving, making conjectures and theorems, algebraic process is integrated in his lessons. This is consistent with the example he remembered during the interview after the second lesson:

> "[…] and we worked with that (activity) for 6 lessons and I showed them how to prove in algebra, about why this must work and I think this was a powerful lesson for them. It was their first instruction in secondary school to algebra and it was very complex but it was answering the questions they had […]".

> "[…] when you try to make algebraic statements the question is always around about how can we be sure this is always the case?".

George also speaks about mathematical community. He obviously prefers the *proof that explains* to the *proof that proves* (Hanna, 2000). He talks about proof as explanation and communication:

> "[…] I believe that the test of proof is 'Does it convince the community?'"

“ […] the mathematical world has set a very high standard of what form this might need to take. In the classroom context it might be less rigorous but is still the issue ‘Does this proof convince the classroom?’”.

3.2. Case study #2: Nicky

Nicky is a less experienced teacher having taught mathematics for two and a half years. She is currently studying for her Masters in Mathematics Education. I observed her doing two lessons with Year 8 students (a different group from George’s Year 8 class) about probabilities. These lessons were the same as George’s.

Concept map

Nicky produced a list of 23 key words and she used 21 words to draw her concept map (diagram 2) although not all were the same with those in the list. Actually, while she described how she drew the concept map she asked herself questions at the same time like: “What ways are there to justify things? How do I convince myself? etc.”. Her central words are clearly “believe” and “need”. She argues that the core function of proof is always to convince yourself and others, even though you can use different methods to explain why a statement is true or not.

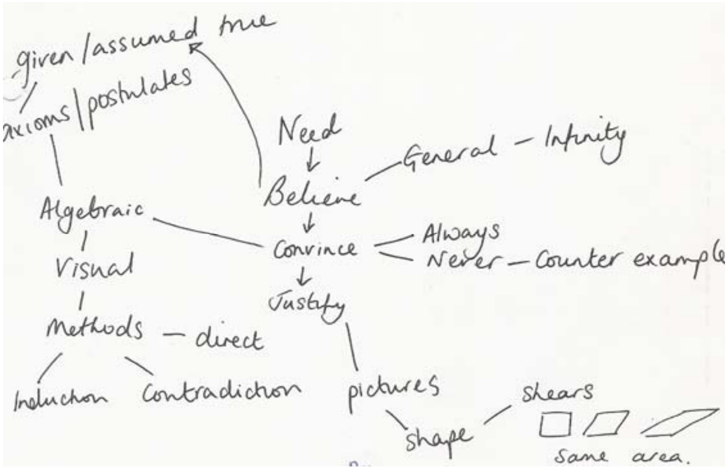

Diagram 2: Nicky’s concept map

She uses the word “algebraic” in relation to her personal experiences:

“[…] often I use algebra to convince myself of something if I am working on maths on my level”.

However, algebraic proof is not necessarily the only way to convince yourself or the others about the truth of a statement. She believes that there are more ways to be convinced such as visual images and diagrams.

Proving task

Nicky thinks Ben’s statement is a good proof because it is general:

“He talked about all cases, so he has talked about everything, so this is general, he is giving a general argument about why this got to be 27, so he is convincing.”

She marked this answer with 10/10 because Ben “shows he understands what it means to prove” and because “his method would work for any numbers whereas the other methods would not”. Consequently, Nicky gave 8/10 to Amina because “she only tried some examples and not all the different ways" and convinced herself but not everybody else. Amina could not find a counter example and her method was exhaustive but she got the second highest mark because she has done considerably more work on the problem than Carol (1/10) and Davor (4/10) who did not understand the problem.

Lessons

Nicky feels that the probability game easily convinced her students and she was happy that she did not have to spend much time on that. The whole classroom discussions were fruitful and everyone was involved, asking intelligent questions. She liked that some clear explanations came from her students and she usually wrote on the board any conjecture:

> "Look at the graph. Do you think this is a good picture of the probabilities? Why this bar is bigger than the others?"
>
> "This is Carly's *conjecture*: If you had two 8-sided dice would there be 64 possible outcomes from adding the totals together? Would a 9-sided dice have the same *pattern*/ graph?"
>
> "There are 6 ways to get 7's so I *expect* the bar to be bigger. The more times you play the more times you expect to get a triangle. This is your *theoretical probability* and you expect this pattern […]"
>
> "There are 15 ways to get 2 6's. Is anyone not *convinced* of what she said? She *proved* her answer."

She is trying to create a culture where all of the students want to prove their conjectures and convince themselves and everyone else in the class:

> "[…] what usually happens is somebody disprove it by giving a counter example or prove it by giving a very *clear explanation and convincing everybody else* in the class […] I tend to talk about proof in the context of their *own conjectures* they come up with […] I hope that there is always a space in the class to prove whatever problem is."

Informal methods "may help students develop an inner compulsion to understand why a conjecture is true" (Hoyles, 1997, p. 8). Therefore, compared to George's conceptions, she seems more detached from the formal idea of proof in the school context. Like many teachers (see Martin & Harel, 1989) her description of formal proofs is very ritualistic in nature, tied to prescribed formats and the use of particular language. Nicky also uses the word "need" which shows that justification in terms of personal convincing is the primary function of proof for her.

3.3. Comparison between George and Nicky

Vollrath (1994) claims that judgments by teachers influence students' appreciation of a theorem. George's response to proof is *affective* (beautiful, surprising, interesting) and Nicky's is *cognitive* (special case, inference) and obviously their explicitly and implicitly expressed views affect students' reasoning skills. Teachers should be aware of their mathematical language and “try to balance the different aspects of knowledge, usage, beauty, culture” (Vollrath, 1994, p. 360).

Both teachers set, as major priorities, classroom discussions and questioning. Furinghetti & Olivero (2001) underline the value of collaborative work and indicate the need for children to share, compare and exchange ideas through discussions. Also, Balacheff (1999) argues that the classroom as a scientific community can be an effective way of making room for proof in school mathematics. Such a classroom environment encourages deductive reasoning. On one hand, Nicky provides her students with experiences of more informal methods of proof and opportunities to formulate and investigate conjectures. On the other hand, George wants his students to be “always wanting to know why something works and have an interest in trying to prove it”. He asks interesting questions that lead students to make and prove conjectures so he claims that his students produce the proofs. However, according to Herbst (2002, p. 198), if students fail to come up with the statement of a conjecture the teacher would have doubts whether this is due to their lack of reasoning skills or the teachers’ failure to provide a fair task. Nicky and George work with experimenting, conjecturing and testing in their lessons – elements necessary to create a classroom culture where proof is always involved.

Results showed that teachers’ existing conceptions of proof have some consistencies and some inconsistencies with their practices. Both teachers hold similar beliefs about the core function of proof – “convince myself and others” – although they are different in the ways each of them reach conviction. George seems more dedicated to the formal and public aspect of proof in his class, whereas Nicky accepts several forms of justification which can satisfy pupils; personal doubts about the truth of a statement (see diagram 3 below).

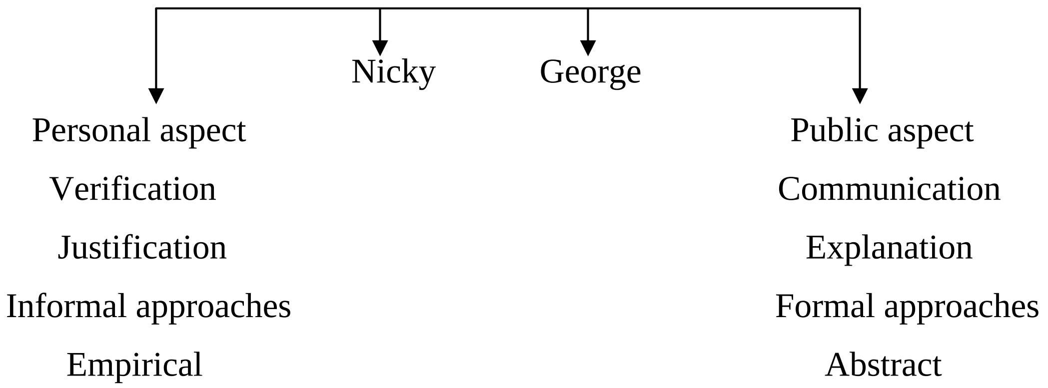

Diagram 3: Comparison between teachers’ beliefs of proof

In other words, George can be characterized as serving proof in the function of communication and explanation and Nicky as serving proof in the function of justification and verification. Chen & Lin (2002) would characterize Nicky’s pedagogical views about teaching proof as a mixture of *convincing-formal view* and *instructional explanatory view*. This means that a teacher convinces students of the truth by manipulation, special cases and demonstrates some kind of explanation. George has a *discursive explanatory view* where the explanation results from students' discourse.

4. EDUCATIONAL IMPLICATIONS AND CONCLUSIONS

Knuth (2002) suggests that implementing "proof for all" might be difficult for teachers. Teacher training programmes and curriculum planners should prepare teachers to teach mathematical proof in the school context, bearing in mind three important elements: a) the levels of proving; b) the functions of proof and c) the approaches to proof.

The comparison between the two teachers reveals the issue of the taught curriculum. Obviously the national guidelines about proof are the same for all teachers; however students do not receive the same instruction. For example, George and Nicky have different teaching approaches based on their beliefs about what proof is. There are also some other factors which influence their performance such as their content knowledge, students' attainment levels, the school and classroom environment and the social context. Consequently, the same material is taught differently and students do not gain the same understanding of the concept of proof.

In conclusion, I highlight the fact that the stronger influence on the relationship between teachers' beliefs and practices does not derive from their past lives and the ways they have been taught; it is teachers' engagement in practical inquiry and their experiences in the classroom (Franke *et al*., 1998) which forms their teaching. Therefore, teachers need to select the most successful methods and prepare effective tasks that respond to the demands of the students and promote *mathematical enculturation* (Bishop, 1988).

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