STRUCTURAL RELATIONSHIPS BETWEEN ARGUMENTATION AND PROOF IN SOLVING OPEN PROBLEMS IN ALGEBRA

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This paper concerns a work-in-progress study analysing cognitive continuities and/or distances between argumentation supporting a conjecture and its proof in solving open problems in algebra. There is usually a cognitive distance between these argumentations and algebraic proofs, not only in the structure (algebraic proofs are often characterised by a strong deductive structure) but also in the "content". The aim of this paper is to show this cognitive distance and the role of abductive argumentation to decrease this distance. Toulmin's model is used as a tool to analyse and compare the structures of argumentation and proof.

INTRODUCTION

This paper analyses cognitive continuities and/or distances between argumentation supporting a conjecture and its proof in solving open problems in algebra. This study, developed as part of the ReMath project (IST - 4 - 26751), can be considered as an extension of a previous research work, studying the relationships between argumentation supporting a conjecture and its proof in solving open problems in geometry (Pedemonte, 2002).

Argumentation supporting a conjecture, developed during the resolution process of an open geometrical problem is often characterized by abductive structure which sometimes remains present in the subsequent proof (Pedemonte 2002). Some experiments highlighted that this structural continuity between abductive argumentation and "abductive proof" does not help students to construct a deductive proof. On the contrary, this "natural" continuity can be considered one of the possible troubles met by students in the construction of a proof.

My research interest is in studying the possibility to extend these research results to other mathematical domains. In particular, in this paper, I consider the resolution processes of an open problem in algebra asking for producing a conjecture and constructing of a proof. The aim of this analysis is to see if there is a "natural" structural continuity between argumentation and proof, which can be considered as one of the possible difficulties met by students in the construction of an algebraic proof.

To perform this analysis I put forward a case study. The experiment has been carried out with students of Formation Science University in Genoa. In this paper, two students resolution processes are presented; their argumentations and proofs are analysed by means of Toulmin's model.

COGNITIVE CONTINUITY AND/OR DISTANCE BETWEEN ARGUMENTATION SUPPORTING A CONJECTURE AND PROOF

Some research studies about argumentation and proof highlight the continuity that exists between argumentation as a process of statement production and the construction of its proof; what is in play is the relationship between conjecturing and looking for a proof (Boero, Garuti, Mariotti, 1996). This continuity is called *cognitive unity*. During a problem solving process, an argumentation activity is usually developed in order to produce a conjecture. The hypothesis of *cognitive unity* is that in some cases this argumentation can be used by the student in the construction of proof by organising in a logical chain some of the previously produced arguments.

Experimental research about cognitive unity (Boero & al., 1996; Garuti & al. 1996; Garuti & al. 1998; Mariotti, 2001) shows that proof is more "accessible" to students if an argumentation activity is developed for the construction of a conjecture. The teaching of proof, which is mainly based on "reproductive" learning (proofs are merely presented to students, they do not have to construct them) appears to be unsuccessful. A didactical consequence of this study is that suitable open problems (Arsac, Germain & Mante, 1991) which call for a conjecture could be used to introduce the learning of proof.

Contributing to this research, a theoretical framework has been developed (Pedemonte, 2002) to analyse and to compare argumentation supporting a conjecture and its proof in solving open problems in geometry. This comparison may be carried out by analysing the continuity or the distance between this argumentation and its proof under two points of view: the *referential system* (Pedemonte, 2005) and the *structure* (Pedemonte, 2007). By *referential system* I mean both the representations system (the language, the heuristic, the drawing) and the knowledge system (conceptions, theorems) of argumentation and proof. By *structure* I mean the logical cognitive connection between statements (abduction, induction, or deduction). For example, there is continuity between argumentation and proof in the referential system if some words, drawings, theorems used in the proof have been used in the argumentation supporting the conjecture. There is a structural continuity between argumentation and proof if some abductive steps used in the argumentation are present also in the proof. Otherwise, if argumentation structure is abduction and proof is deduction there is a structural distance between the two.

Research results carried out by this study (Pedemonte, 2007) highlight the importance of structural analysis between argumentation and proof. This analysis shows that although there are cases of continuity in the referential systems between argumentation supporting a conjecture and its proof, it is often necessary to cover a structural distance between the two (from an abductive argumentation to a deductive proof). This structural distance is not always covered by students, who sometime produce incorrect proofs because they are not able to transform the structure of argumentation in deductive structure for proof (Pedemonte 2007).

These research results are limited to the geometrical domain, which is the mathematical domain where usually learning of proof is introduced. Nevertheless, it could be interesting to analyse if it is possible to extend such results to other mathematical domains. In this paper, algebra is considered.

THE ROLE OF ALGEBRA IN PROVING PROCESS

The solution of open problems in algebra asking for a conjecture seems to be usually characterized by two particular phases: the *constructive argumentation* (Pedemonte, 2002) phase that corresponds to the construction of a conjecture (sometime only characterized by numerical examples); the proof phase that concerns the systemic application of algebraic rules, in which each step of the proof is the transformation of the previous step according to a given rule. During the resolution process it is possible to produce another type of argumentation, the *structurant argumentation* (Pedemonte, 2002), which is constructed to justify a conjecture, in particular when the conjecture is constructed as a "fact". I think that this argumentation can play an important role in the resolution process of open problems in algebra.

As a matter of fact, some cognitive research about the resolution of algebraic problems (Duval, 2002) highlights the cognitive gap between the *conversion phase* (or the constructive argumentation phase), i.e. the translation of the problem in algebraic characters, and the *treatment phase* (or the proof phase), i.e. the deduction of the unknown value of the algebraic expression. According to Duval, this gap has to be coped with by students in solving problems in algebra.

Moreover, if we consider that sometime argumentation in open problems in algebra is characterized by explorations based on arithmetic numerical examples, the gap between constructive argumentation and proof is also present as methodological aspect (Chevallard, 1989). Arithmetic moves from known to unknown while algebra often moves from unknown to known in way that at the end of the process it is always possible to know the unknown quantity. Arithmetic and algebra have two separate languages: the first one is based on the ordinary language enriched by a numerical language while the second one is essentially oriented to computation where there is a mechanic control. Other research studies highlight difficulties in catching the invariance of algebraic denotation respect to the sense (Arzarello & al., 1994, Drohuard, 1992); in arithmetic this invariance is automatic because denotation is a specific number while in algebra it is connected to the syntactic aspects.

Following this research results, I make the hypothesis that *structurant argumentation* could be useful to decrease the cognitive gap between the *constructive argumentation* and the proof. In particular, a successful structurant argumentation should favour the continuity in the referential system between constructive argumentation and proof.

From a structural point of view my hypothesis is that in solving open problem in algebra, the “natural” structural continuity between argumentation and proof is usually not present. The connection between two steps in algebra is characterized by a “strong” deductive structure: algebraic expression as equations are modified according to computation rules often implicit for student. This is not always the case for argumentation supporting a conjecture. Argumentation structure can be abductive, or inductive if conjecture is constructed as generalisation on numerical examples.

The problem is that the structural distance between argumentation and proof contributes to increase the gap between the two and sometime student is not able to reconstruct reasoning used to construct the conjecture.

This hypothesis will be illustrated in the next section, where I present the analysis of two student protocols. To complete this discussion, I am going to introduce Toulmin’s model as a tool to analyse proving process performed by students.

Toulmin’s model: a methodological tool to analyse argumentation and proof

As methodological tool to analyse and to compare argumentation and proof I use Toulmin’s model (Toulmin, 1958). In this model argumentation, as proof, has a ternary structure. This fact allows us to compare the structure of argumentation with the structure of the proof.

In Toulmin’s model an argument comprises three elements (Toulmin, 1958/993):

C (claim): the statement of the speaker,

D (data): data justifying the claim C,

W (warrant): the inference rule, which allows data to be connected to the claim.

In any argument the first step is expressed by a standpoint (an assertion, an opinion). In Toulmin's terminology the standpoint is called the claim. The second step consists of the production of data supporting the claim. The warrant provides the justification for using the data conceived as a support for the data-claim relationships. The warrant, which can be expressed by a principle, or a rule, acts as a bridge between the data and the claim.

The basic structure of an argument is presented in Figure 1.

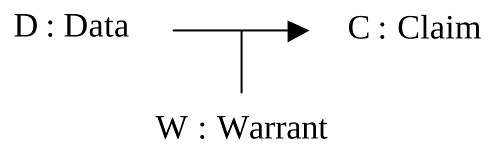

Figure 1: Toulmin’s basic model

Three auxiliary elements may be necessary to describe an argument: a qualifier, a rebuttal, a backing (Toulmin, 1958/993). These elements are not significant for the analysis treated in this paper and for this reason will not be presented.

In Toulmin's model a step appears as a deductive step: data and warrants lead to the claim. Nevertheless, it could be useful to represent other argumentative structures using this model. In this paper we consider abductive structure.

Abduction has been introduced by Peirce (Peirce, 1960) as a model of inference used in the discovery process. According to Peirce, starting from an observed fact, a rule can be supposed, through which the hypothesis becomes more credible. The hypothesis is the conclusion of a reasoning giving it a plausibility value (Peirce 1960, 2.511n). So abduction is a plausible reasoning (Polya, 1962) which can be modelled as follows (Polya 1962, p. 107):

If A then B
B true
____
A more credible

By this scheme we can represent an abductive step in Toulmin's model as follows:

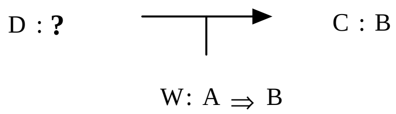

Figure 2: Abductive argumentation in Toulmin's model

The question mark means that data are to be sought in order to apply the inference rule justifying the claim.

Drawing on Toulmin's model (Toulmin, 1958), I analyse structural continuities and structural distances between argumentation and proof.

CASE STUDY

In this section two resolution processes of an open problem in algebra are presented. They are taken from a set of data collected with prospective primary school teachers attending a math course at the University. Students were asked to solve the problem aloud, they worked alone under the supervision of a researcher who do not intervene in helping them. The observation was conducted out of the usual schedule. Students' mathematical background was not homogeneous because students came from different schools. Nevertheless, all of them could solve the problem with their theoretical algebraic background, even if they were not familiar with problems of this kind.

The problem presented to students is the following:

"What can you say about $(p-1)(q^2 -1)/8$ if p and q are odd numbers?"

This is a classical problem, analysed by different research studies (Arzarello & al. 1994, Garuti & al. 1998).

I transcribe the main part of two resolution processes which are based on the transcriptions of the audio recordings and the written productions of the students.

Two examples are presented:

- Example 1: Example of structurant argumentation which decreases the gap between constructive argumentation and proof
- Example 2: Example of structurant argumentation which does not decrease the gap between constructive argumentation and proof

In order to analyse the argumentation, I have selected the assertions produced by students and reconstructed the structure of the argumentative step: claim C, data D and warrant W. The indices identify each argumentative step. The student's text is in the left column, and my comments and analyses are reported in the right column. The texts have been translated from Italian into English.

Example 1

Manuela constructs the conjecture as generalization of numerical examples. The structure of this argumentation is inductive and the referential system is based on arithmetic.

If p=11 and q=13 then …*(She calculates)* the result is 210 If p=7 and q=9 then the result is … 60 They are even numbers Then probably $(p-1)(q^2-1)/8$ is an even number	**probably** **D_1: Previous arguments** → **C_1: The expression $(p-1)(q^2-1)/8$ is an even number** **W : generalisation** *Conjecture is a fact constructed as generalization on numerical examples; now student has to justify it.*

Manuela produces a structurant argumentation to justify her conjecture. She analyses expression $(p-1)(q^2-1)/8$ considering even and odd numbers properties. She is not able to conclude.

if p is an odd number, p-1 is even; if q is an odd number, q^2-1 is an even number too, Then an even number times an even number is an even number, then the expression is an even number…	**D_2: p is an odd number** → **C_2 : p-1 is an even number** **W : even and odd number property** **D_3: q is an odd number** → **C_3 : q2 -1 is an even number** **W : calculus rule and even and odd number property** **D_4: C_2, C_3** → **C_4 :$(p-1)(q^2-1)$ is an even number** **W : the product of two even numbers is an even number**

Manuela understands that the claim C_4 is not sufficient to justify the conjecture. She looks for another element allowing her to state the conjecture.

… but there is the division by 8. I have to find something else to say that the expression is even.	**D_5: C_4 and ?** → **C_5 = C_1: $(p-1)(q^2-1)/8$ is an even number** **W : calculus rule and even and odd number property** *This is an abductive structure.*

The last argument is very important in the structurant argumentation because it leads Manuela to look for something else (the question mark in D_5) to justify the conjecture. She analyses q^2-1. By some numerical examples Manuela understands that q^2-1 cannot be less than 8. She does not say explicitly that q^2-1 is divisible by 8 but we can suppose she makes this consideration because she considers different values for q (1, 3, 5, 7, 9)

But q^2-1 cannot be equal to 2 neither 4 because with 1 q^2-1 is 0 and with 3 q^2-1 is 8. Then the minimal number is 8; 8 over 8 is 1 then the expression is an even number.	**D_6: Substitution in q2 -1 of some numerical examples (q=1, q=3)** → **C_6: q^2-1 cannot be equal to 2 or to 4** **W : calculus** **D_7: C_6** → **C_7 : q^2-1 cannot be less then 8** **W : comparison among numerical substitution in the formula** *Manuela can complete data 5 by the claim C_7.*

We observe that the structurant argumentation is characterised by both arithmetic and algebraic reasoning. Manuela looks for elements useful to construct the proof.

q = 2n+1 then q^2-1 = $4n^2$+4n+1-1 = 4n(n+1). This is at least divisible by 4, and so what remains is n(n+1), which is surely divisible by two, because if n is even everything is fine, if n is odd (n+1) is even and then 4n(n+1) is at least divisible by 8. We may conclude that q^2-1 is a multiple of 8. Then $(p-1)(q^2-1)/8$ is even if p and q are odd.	**D_8: q = 2n+1** → **C_8: q^2-1 = 4n(n+1)** **W : substitution** *Manuela analyses the formula 4n(n+1) to prove that q^2-1 is at least 8.* **D_9: n is an even number** → **C_9: 4n(n+1) is at least divisible by 8** **W : calculus rule (4*2t=8t with 2t=n)** **number** … **least divisible by 8** **W : n+1 is an even number and calculus rule**

Arguments 9 and 10 allow concluding that 4n(n+1) is always at least divisible by 8. Then the argument 5 is transformed into a deduction step and the conclusion is carried out rapidly. We can observe that referential system is based on arithmetic for the constructive argumentation and on algebra for proof. The structurant argumentation contains arithmetic and algebraic elements allowing the continuity in the referential system between the two. It is the abductive step in structurant argumentation which allows the connection between the constructive argumentation and the deductive proof: Manuela analyses the expression 4n(n+1) to prove that this expression is divisible by 8.

Example 2

Let's consider the answer produced by another student, Elio. He tries different strategies: at the beginning he produces a reasoning similar to the previous one (the arguments 2, 3 and 4 of the previous example) concluding that the expression $(p-1)(q^2-1)$ is an even number. Nevertheless, he says that this fact it is not useful "because in general it is not true that an even number divided by another even number makes an even number". Then he assigns some numbers to the letter p and q and by means of a generalisation he constructs conjecture.

If p=1 and q=3 then 0*8/8=0 p=5 and q=7 then 4*48/8=24 p=11 and q=13 then 10*168/8=210 It seems that the expression (p-1)(q2-1)/8 is even.	**probably** **D_1: Previous arguments** → **C_1: The expression $(p-1)(q^2-1)/8$ is an even number** **W : generalisation** *Conjecture is constructed as generalization based on numerical examples.*

As in the previous example conjecture is based on arithmetic examples. These examples allow concluding that q^2-1 seems to be divisible by 8.

And…. Wait… It seems that by substituting q with an odd number, q^2-1 is divisible by 8. Then $(p-1)(q^2-1)/8$ is even because p-1 is even and q^2-1 is divisible by 8	**D_2: Claims based on numerical examples** → **C_2: q^2-1 is always divisible by 8** **W : generalisation** *Elio concludes that q^2-1 is divisible by 8. Then he justifies the conjecture:* **D_3: p-1 is an even number and C_2** → **C_3: C_1** **W : calculus rules and even and odd number properties**

Elio has constructed a structurant argumentation which allows him to justify conjecture. Nevertheless this justification is still based on arithmetical examples. Moreover, there is no abductive step to connect constructive argumentation with proof. As a matter of fact, Elio tries to construct a proof but without any result. He looses the connection with the argumentation phase; he is driven by deductive structure of algebraic proof.

I try to prove the statement p and q are odd, then p =2k+1 and q=2h+1 then I can find (2k+1-1) [(2h+1) ^{2} -1]/8 2k[(4h^2+1+4h)-1]/8 2k(4h^2+4h)/8 which is not equal to 2 times something… I cannot conclude I can simplify k(4h^2+4h)/4 If I factor 2...2k(2h^2+2h) /4 no... If I factor h: 2kh(4h+4) /8 no… If I factor 4h: 2k*4h(h+1)/8 no I cannot prove with algebra, but I'm sure that the expression is an even number	**D_4: (p-1)(q^2-1)/8 → C_4:(p-1)(q^2-1)/8 = (2k+1-1) [(2h+1)^2-1]/8** **W : Conversion phase by substitution with p=2k+1 and q=2h+1** **D_5: C_4 → C_5:(p-1)(q^2-1)/8 = 2k*(4h^2+4h)/8** **W : calculus rules** **D_6: C_5 is not equal to 2 times something → C_6: I cannot conclude that (p-1)(q^2-1)/8 is an even number** **W : even number property** *In proof Elio performs other transformations of the formula but without any result. The other arguments are similar to the previous and they carry out the same result: Elio is not able to construct the proof.*

Elio has solved the problem but he is not able to construct the proof. The strength of the deductive chain seems to be so strong that Elio is not able to construct *continuity in the referential system* with the argumentation; he manipulates the formula to find an expression of the form "2 times something". He looses the connection with the referential system. We can observe that in this case, structurant argumentation does not produce the connection between arithmetic and algebra; this structurant argumentation is still based on numerical examples. Moreover, there is not an explicit abductive step in structurant argumentation which could help Elio to focus which elements lack to justify conjecture and construct the proof.

CONCLUSION

By means of Toulmin's model, we have analysed two resolution processes of an open problem in algebra. In both cases a structurant argumentation was present. In the first case this argumentation allows the construction of the proof while in the second one this argumentation has not been successfully for the construction of the proof.

At this stage of the research we can conclude that, in solving an open problem in algebra, a structurant argumentation can be useful for the construction of the proof if it favours the continuity between constructive argumentation and proof in the referential system. Moreover, in opposition to the geometrical case, abductive structure in structurant argumentation does not represent one of the possible trouble met by student in the construction of the proof because the strength of deductive structure in algebraic proof prevents at least partially the occurrence of structural continuity between argumentation and proof.

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