PROOFS PROBLEMS IN ELEMENTARY NUMBER THEORY: ANALYSIS OF TRAINEE TEACHERS' PRODUCTIONS

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Abstract: Our study involves a group of pre-service middle-school [1] *teachers attending educational and training courses at our university. The aim is to classify their behaviour in solving a problem that requires the proof of a statement in elementary number theory not easy to be formalized. This study highlights mental blocks in those who are not able to create an algebraic model for the problem and widespread difficulties related to the impact of abilities both in translating algebraic expressions into the algebraic code and in interpreting algebraic expressions built during the construction of the proof in order to get to the thesis.*

INTRODUCTION

The Italian mathematics curriculum's strong influence on students' failures in constructing proofs is unquestionable. Indeed, the Italian teaching tradition focuses on training students to reproduce proofs and does not devote time to students' autonomous construction of proofs. We are convinced that students should be encouraged to construct important mathematical facts and investigate problems starting from compulsory school, especially in the 6^{th}, 7^{th} and 8^{th} grades. In Italy, more than 80% of lower secondary school mathematics teachers are not mathematics graduates, and their cultural background makes it very difficult for them to carry out this kind of activity.

Recent studies suggest that the teaching of the concept of proof should be promoted in pre-high school grades (Stylianides e Stylianides, 2006b). Nevertheless, in order to promote students' proving abilities, it is necessary for teachers to be able to autonomously manage the solution of proving problems. We agree with Jones (1997) when he writes that "the most-qualified trainee teachers may not necessarily have the specific kind of subject matter knowledge needed for the most effective teaching". Therefore, an investigation on the correlation between teachers' educational background , their conceptions and the different approaches they adopt towards proof problems is necessary. Moreover, we need to highlight the main difficulties they meet, so that they can become aware of both potential and limitations of their own mathematical background. In this way, it will be possible for teachers to develop the content knowledge that can allow them to promote argumentation and proof in their classrooms (Stylianides and Stylianides, 2006a).

Therefore, we decided to work with a group of trainee teachers [2] attending educational and training courses for the teaching of mathematics in middle-school: our aim was to study the impact of abilities in formal coding and in interpreting algebraic expressions on proofs' production, with particular relation to the transition between argumentation and proof. Boero et alii (2002) compared argumentation and proof as linguistic products, stressing that the language adopted is one of the most relevant points of discrimination between them. In particular, they observe that "the understanding and production of mathematical proofs belong to the literate side of linguistic performances and require prior deep linguistic competence".

Our hypothesis is that the rupture between argumentation and proof gets deeper due to a lack of activities of both translation from verbal to algebraic language and interpretation.

The analysis we propose here has a double value:

1) It highlights the influence of educational background on the choice of a proving strategy as well as on the problems related to the development of a proof;

2) It represents a moment in a formative "route", proposed to trainees in order to make them: a) understand the importance of coordinating verbal and algebraic language in the development of proofs; b) compare proof strategies developed through both the verbal and the algebraic register; c) reflect on potentialities and limitations of different strategies (Afterwards we will show what kind of theoretical tools we referred to in order to promote trainees reflections on these aspects).

In this paper we focus on a problem posed to trainees. We chose it because of its particular textual characteristics and because of the kind of proof it implies. In fact, its solution could be "easy" to be found from an intuitive point of view, but its formalization is quite complex. Before presenting an analysis of this problem, we will sketch out a synthesis of some studies about the subject of proof.

THEORETICAL FRAMEWORK

Issues related to the meaning of proof and to its functions have been deeply analysed both from the mathematicians' community's and from teachers and mathematics education researchers' points of view (Hanna 2000; Hersh 1993; Thurston 1994).

Investigating students' difficulties in producing proofs and searching for reasons underlying their failures or successful results are considered to be crucial issues (Hoyles 1997; Weber 2001). Some researchers have identified proof schemes through which the different attempts carried out by students in developing a proof can be classified (Alcock and Weber 2005; Harel and Sowder 1998). Harel and Sowder, in particular, subdivide proof schemes into three main classes, each of them further divided in subcategories: *external conviction proof schemes* (produced by students who think that "ritual and form constitute mathematical justification"), *empirical proof schemes* (produced by students who "validate, impugn or subvert conjectures by appeals to physical facts or to sensory experiences"), *analytical proof schemes* (produced by those who are able to "validate conjectures by means of logical deductions").

Other researchers outlined theoretical models for students' difficulties in proving and suggested possible strategies to promote an appropriate attitude by students in dealing with proofs (Moore 1994; Weber 2003).

In the last decade, studies about students' production of algebraic proofs in the elementary number theory field intensified also due to the space given to proof in the curriculum of some countries (England is an example). From a didactical point of view, the construction of algebraic proofs creates problems: in fact, as it is stressed by Barnard and Tall (1997), while algebraic manipulation requires to carry out "sequential procedures in which each mathematical action cues the next", a proof also requires the ability of making choices. Furinghetti and Paola (1997) highlighted the presence of a double "shadow effect" when students face proofs of statements in elementary number theory: a "shadow effect" due to algebra, which prevents students from using their arithmetic knowledge in their attempts of proving, and a "shadow effect" due to arithmetic, which leads students to view only numerical checks as proofs and prevents them from making generalizations explicit.

Analysis of students' proofs of statements in elementary number theory highlighted students' difficulties not only in translating familiar numerical concepts (such as "to be even" or "to be odd") from verbal to algebraic language, but also in deducing all the possible information that an algebraic expression brings with it (Barnard and Tall, 1997).

Some mathematics education scholars propose an approach to algebraic language also related to the development of reasoning in the proving process, referring to natural numbers as a suitable environment for those activities that favour a transition from argumentation to proof through the use of algebraic language (Boero & al. 1995; Malara 2002; Friedlander, Hershowitz, Arcavi, 1989; Sadovsky, 1999).

PRESENTING THE PROPOSED PROBLEM

The problem at stake is the following: *"Suppose that a is a non null natural number. If a is divisible neither by 2 nor by 3, then* a^2*-1 is divisible by 24"*. This problem is taken from the textbook, aimed at 15-16 years old students, "Matematica come scoperta" ("Mathematics as discovery") by G. Prodi (1979). This textbook was thought and written in a research-based environment and it is still very innovative.

We made hypotheses about possible difficulties related to the interpretation of the problem's statement and to the choice of the proof strategy.

As regards difficulties related to text interpretation, we point out that this problem is different from those typical tasks students are exposed to because of its linguistic formulation: the hypothesis is, in fact, expressed by a negation (non-divisibility by 2 and 3). Another element of difficulty in the interpretation of the text is related to the fact that the thesis refers to an element (a^2-1) different from those the hypothesis "talks about" (a).

Other difficulties depend on the choice of the proof strategy. A verbal "approach" [3] to the proof of this statement is based on considerations that concern the concept of (non) divisibility by 2 and by 3 and on considering the relation that involves the dividend, the divisor, the quotient and the remainder of a division. It requires a good control of the implications of the hypothesis, in particular a clear view of the properties of both quotient and remainder of the division by 2 and by 3. The verbal proof develops from the identity $a^2-1=(a-1)(a+1)$: i. ($a-1$) and ($a+1$) are even because a is not-divisible by 2; ii. one among ($a-1$) and ($a+1$) is also divisible by 4 because they are two consecutive even numbers; iii. one among ($a-1$) and ($a+1$) is divisible by 3 because *a* is not-divisible by 3; therefore iv. $(a-1)\cdot(a+1)$ is altogether divisible by 24.

Through an *algebraic* approach one can be led to consider also the formalization of the relation between the terms involved in the division of *a* by 6, besides the formalization of the similar relations that can be inferred from the division of *a* by 2 and by 3.

The possible algebraic proofs are more complicated than the verbal one because they require the formal translation of the hypothesis, a distinction between different cases, the use of suitable syntactic transformations and, above all, the interpretation of the new expressions produced with relation to the problem.

Therefore we can imagine that an algebraic "approach" to this problem is likely to bring about obstacles for students, in particular related to the translation of divisibility properties in terms of the Euclidean algorithm. Control of the syntactic equivalence of expressions having different senses is fundamental, because it allows one to highlight properties that emerge more clearly in some expressions than in others (Arzarello, Bazzini, Chiappini, 1995).

The choice of this problem is related to the fact that an algebraic "approach" to its solution makes it possible to highlight trainees' flexibility in both formalization and interpretation of formal expressions and also their possible mental blocks.

METHODOLOGY

The problem was given to 54 trainees with different university backgrounds (27 mathematics graduates, 3 physics graduates and the remaining 24 biology, geology, chemistry and natural sciences graduates) in the initial phase of the Mathematics Education training courses. Trainees were supposed to solve, in 45 minutes, the assigned problem, describing the different proving strategies they tried to follow in the solution process and pointing out both obstacles and difficulties they met.

In analysing their protocols, we divided trainees into two groups according to their backgrounds: mathematics or physics graduates and trainees with a degree in other sciences. Our analysis is made through a double lens. In fact, we look at trainees' protocols as products, classifying the proofs they gave making reference to Harel and Sowder (1998)'s classification of proof schemes. At the same time, we analyse their protocols from the point of view of the difficulties which come out in the proving processes. In this second analysis we look at difficulties caused by: problems related to the coordination between the verbal and algebraic registers in carrying out the proof; inability to formalize compound statements; inability to interpret formulas resulting from a syntactic transformation; presence of possible mental blocks.

RESULTS

We divided trainees' productions into six main groups according to the highlighted proving strategies and to Harel and Sowder's proof schemes (we singled out those categories which adhere to the examined proofs). The following table summarizes the proof schemes' frequencies in our trainees' protocols.

Table 1: Frequencies of proof schemes with relation to educational backgrounds

Harel and Sowder's Classification		Typologies of protocols which adhere to categories	Frequencies of proof schemes with relation to educational backgrounds	
Categories	**Subcategories**		Maths-Physics	Other sciences
External conviction proof schemes	*Symbolic proof schemes*	Algebraic proofs which highlight syntactical and/or interpretative problems in managing formal expressions	**9/30**	**8/24**
	Ritual proof schemes	Proofs in which trainees attempt to apply well-known proving procedures (such as reasoning by induction and proof by contradiction) characterised by serious logical mistakes	**3/30**	/
Empirical proof schemes	*Perceptual proof schemes*	Verbal proofs in which reference to erroneous perceptions about numerical properties is made explicit	**2/30**	**2/24**
	Inductive proof schemes	Proofs based on numerical examples only	**1/30**	**14/24**
Analytical proof schemes	*Intuitive-axiomatic proof scheme*	Verbal proofs in which reference to correct numerical properties is made explicit	**2/30**	/
	Symbolic transformational proof schemes	Correct algebraic proofs	**13/30**	/

The table highlights that *Trainees with "other scientific degrees"* definitely prefer verification through numerical examples (14 out of 24). Those who try to use the algebraic code (only 8 out of 24) get stuck because of their difficulties in manipulating and understanding the algebraic expressions they formulate. The remaining trainees with "other scientific degrees" try, not successfully, to prove the statement using a verbal approach. Those who try to construct a not merely inductive proof also turn to numerical examples. In fact they use numerical examples in their proofs when they want to show the correctness of their assertions or when they are not able to manipulate some expressions. In this context, this attitude hides insecurity. With reference to trainees who verify the statement through numerical examples only, we think it is important to group their protocols according to whether they show consciousness about the limits of their productions or not. In this regard, we can say that most trainees (10 out of 14) are aware that numerical examples are not enough to prove the statement, so they point out their inability in finding an algebraic representation of the regularities observed through examples. Four trainees, however, think that verifying a property for some numerical cases only is enough to prove it.

As regards "aware" trainees, these are some significant claims made by some of the trainees after having shown that the property holds for some simple numerical examples: *"I can guess that it's true, but I'm not able to formalize it in mathematical terms, so I'm not able to prove it"; "I can't express numbers through symbols"*.

As regards "not-aware" trainees, there are some important claims made by trainees who believe they have correctly proved the statement: *"the property is valid when a=5, 7, 11 so, if we substitute a for 13, we'll find a multiple of 24"*.

Almost all *mathematics and physics graduates* (25 out of 30) prefer an algebraic "approach". Four of them try to give a verbal proof (2 unsuccessfully). Only one of them chooses the inductive scheme. We were negatively surprised by the high number of trainees with a degree in mathematics and physics who failed in the proof of the statement (15 out of 30; 12 of them chose the algebraic strategy) because of their inability to organize a proof strategy or to complete the proof.

We observe the worrying fact that a high number of trainees are not able to master algebraic expressions (41 out of 54, precisely all the trainees with "other scientific degrees" and 17 out of 30 trainees with a degree in mathematics or physics). Some of them openly declare their difficult relationship with algebraic language, others display a rejection of mathematics when they choose not to use any algebraic formalism, others display their difficulties through their unsuccessful attempts to master the algebraic expressions they formulate.

A ZOOM ON PROVING PROCESSES

In the following we will show the <u>main difficulties highlighted by our analysis</u> of trainees' protocols, distinguishing between trainees with a degree in mathematics or physics and trainees with other scientific degrees. We will start from the latest because of their greater weakness in the development of formal reasoning.

Trainees with "other scientific degrees"

Among the main problems highlighted by our analysis of protocols, we would like to stress two kinds of difficulties in particular:

1) <u>Difficulties in correctly interpreting the text of the problem and in inferring properties related to hypothesis and thesis</u>. These are some of the typical mistakes:

1a) A lot of trainees draw wrong conclusions about the set of numbers which satisfy the properties required for *a*. For example, a widespread interpretation makes some trainees conclude that *a* must be a prime number;

1b) Some trainees display that they do not know exactly what to prove. A trainee, for example, writes confidently that it is impossible to prove the statement because *"Every number which is divisible by 24 is also divisible by 2 and by 3"*, confusing the properties of *a* with the properties of a^2-1.

2) Difficulties in interpreting and managing algebraic expressions. Typical mistakes are related to the process of formal coding and to the interpretation of expressions after the syntactic treatment. These are some examples of the most representative mistakes, that we propose following the development of the proof:

2a) Those who try to factorize a^2-1 stop after having observed that $(a-1)(a+1)$ is divisible by 2 and by 3 and they are not able to infer other simple properties, such as the divisibility of $(a-1)(a+1)$ by 4.

2b) Almost no trainees are able to formulate the properties of *a* in algebraic language, and in particular the non-divisibility of *a* by 3. Some of them try to solve the problem interweaving algebraic and verbal language. One trainee, for example, expresses the non-divisibility of *a* by 2 and by 3 in this way: *"a/2?integer and a/3?integer for every integer greater than or equal to zero"*. Others translate the properties of *a* into expressions with a completely different meaning, as we will see afterwards in a reported example.

2c) Those who try to prove the statement show that they do not know how to deduce the properties of a^2-1. One trainee, for example, starts his proof with the formalization of the thesis and writes $a^2-1=24n$. He immediately stops here because he is not able to further interpret what he has written applying the hypothesis.

In support of the latter remarks related to difficulties in the interpretation of algebraic expressions, we propose an *example of an unsuccessful attempt to give a formal proof, not supported by a real understanding of the algebraic expressions*:

The trainee observes, without any motivation, that every number which is divisible neither by 2 nor by 3 can be written as $(3+2)+n$, where *n* is an even number. Afterwards, he carries out some calculations, writes $(3+2+n)^2-1=24+10n+n^2$ and, in order to prove that this expression is divisible by 24, he sets up the equation $24+10n+n^2=24h$. Then he manipulates the equation, obtaining the equality $n=24\frac{h-1}{10+n}$. Finally, he uses this last equality to assert that *"n is divisible by 24"*.

Among the observations that we can make about this proof, we think it is important to stress the total lack of control in understanding and managing both the algebraic expressions that he constructs and the related properties. He is not able to translate the properties of *a* into formal language; he sets the condition that the expression obtained for a^2-1 satisfies the required property, confusing hypothesis and thesis; he obtains an equality in which *n* is not even made explicit and draws from it erroneous conclusions about the properties of *n*; he draws conclusions about *n* instead of drawing conclusions about a^2-1.

Trainees with a degree in mathematics or physics

Most of those trainees who fail in their proof display: 1) difficulties in the translation form verbal to algebraic language, closely connected with a clear lack of knowledge about elementary numerical properties; 2) difficulties in understanding and managing formal expressions.

With reference to point 1, the main mistakes in translation are related to trainees' inability to formalize concepts such as multiple or divisor, and to use the Euclidean relation between dividend, divisor, quotient and remainder of a division:

1a) Some trainees are not able to make the hypothesis explicit. The following claim clearly shows this problem: *"It's difficult to codify in symbols the concept of not being divisible by 2 and not being divisible by 3"*;

1b) A lot of trainees try to use the Euclidean relation for representing the non-divisibility of a by 3 or by 6, but they do not consider the variability of the remainder. For example, some of those who deduce from the hypothesis the non-divisibility of a by 6 translate this property into the equality $a=6n+1$ only;

1c) A small number of trainees, after factorizing $a^2-1=(a-1)(a+1)$, are not able to correctly deduce the properties of the factors $a-1$ and $a+1$ from the hypothesis. One trainee, for example, erroneously deduces that *"a-1 and a+1 must be two odd numbers because a is non-divisible by 2"* and erroneously claims that *"a-1=2h+1 and a+1=2h+3, with h?1 (mod3) e h?0 (mod3) because a is not divisible by 3"*. He attaches characteristics of a to $(a-1)$ and $(a+1)$ and imposes not-justifiable conditions.

With reference to point 2, we observed that many trainees fail because:

2a) Even if they correctly set up the proof, translating, without any difficulty, the properties of a, they get stuck during the manipulation of the algebraic expressions they formulated. For example, a trainee correctly expresses a as $a=3h+1$ (h even) or as $a=3h+2$ (h odd) and, performing syntactic elaborations (substitutions and transformations), he obtains, in the first case $a^2-1=12n(3n+1)$ and in the second case $a^2-1=36n^2+24n+3$. However, he is not able to proceed in order to reach the thesis.

2b) Even when they are able to carry out syntactic elaborations, they do not always correctly interpret the new expressions they obtain. One trainee, for example, considers the case $a=6k+1$ and obtains, through syntactic transformations, the expression $(6k+1)^2-1=12k(3k+1)$, but he concludes that a^2-1 *"is divisible by 12 and not by 24"*, without considering that k or 3k+1 could be even. In this case, we can hypothesize that the standard representation of an odd number influences his erroneous conclusion.

The protocols of some trainees deserve a separate remark because they represent ruinous attempts to apply well-known proving procedures. These protocols highlight, in some cases, a total lack of understanding of the meaning of such procedures and, in other cases, serious mistakes in logic. A trainee, for example, attempting to prove the statement, proves the inverse proposition ("if a^2-1 is divisible by 24, then a is divisible by 2 and by 3"), highlighting a serious gap in logic because he thinks that proving the truth of the inverse of a proposition ensures that the original proposition is true.

FINAL CONSIDERATIONS

A high number of trainees with a degree in mathematics or physics had remarkable difficulties and gave only partial solutions, full of gaps; this fact, together with the general negative results of trainees with other scientific degrees leads us to state that: 1) those who were not appropriately educated to translate verbal expressions into the algebraic code either avoid the use of algebraic language or, in the attempt to use it to develop a proof, are unable to interpret and manage the formal expressions they formulate; 2) those who are well-trained in the syntactic manipulation of algebraic expressions get stuck if they are not appropriately educated to interpret them.

Therefore, it is essential to educate both students and trainees not only to be able to translate, but especially to interpret. It is clear that, if pre-service teachers do not master the proving activity, they are probably not able to propose it to their students in a persuasive way. Hence, we think that it is important to put the stress on proof during educational and training courses for teachers, as regards both abilities to attain a proof and the awareness of the role it plays in the mathematical activity. This "investment" is more urgent for those who are going to teach mathematics without having a background in mathematics.

In the future development of this study we want to work with teachers also on the comparison of their productions, in order to favour the construction of proofs as well as a continuous reflection about both the related difficulties and the feasibility of such activities in the classroom. In this sense, we think it is important to give them some theoretical instruments that could be a good reference in order not only to facilitate the communication between trainees and trainers, but also to make teachers aware of how to "move around in the proving universe".

NOTES

1. In Italy middle-school corresponds to grades 6^{th}, 7^{th}, 8^{th}.

2. Pre-service teachers attending SSIS (Specialisation School for secondary school teaching) at the University of Modena and Reggio Emilia.

3. By "approach" we mean "way of putting oneself in front of a problem in relation to the choice of a particular linguistic code".

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