INDIRECT PROOF: AN INTERPRETING MODEL[§]

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Starting from a general discussion on mathematical proof, a structural analysis was carried out, leading to the construction of a model within which indirect proofs can be described. The model shows itself a good interpreting tool to identify and explain cognitive and didactic issues, as well to precisely formulate research hypotheses concerning students' difficulties with indirect proofs.

INTRODUCTION

Mathematics education literature offers a rich and varied panorama of theoretical frameworks within which different didactic issues related to proof were identified and studied (see for instance Balacheff, 1987; Duval, 1992-93; Harel et al., 1998; Garuti et. al.,1998; Mariotti et. al., 1997; Pedemonte, 2002). However, the research studies carried out within these frameworks only rarely addressed issues that can be related to some specific logical structure of a proof. Some authors focused their attention on proof by mathematical induction (Harel, 2001; Pedemonte, 2002), not much attention was devoted to indirect proofs, with which, on the contrary, this paper intend to deal.

Although not always easily comparable, research studies devoted to indirect proof report consistent results concerning students' difficulties with this type of proof, that, at any school level, seem to be greater than those related to direct proof. Different interpretations and different sources of these difficulties were proposed. Certainly, as some authors remarked, indirect proofs do not find an adequate attention in school practice, at any school level (Thompson, 1996; Bernardi, 2002). But, although correct, this remark cannot fully explain the difficulties that students seems to face. Taking a cognitive perspective, other studies contributed to identify specific aspects that, affecting students' cognitive processes, could be responsible of their difficulties. For instance, assuming and dealing with false hypotheses was recognized as one of the main sources of difficulties (Leron, 1985). As far as the production of a proof is concerned, Wu Yu et al. (2003) and Antonini (2001, 2003a) demonstrated how difficulties could be found at the very beginning, when there is the need of correctly formulating the negation of statement. The distinction between different functions of a proof led to interpret students' difficulties (Barbin, 1988), to analyse conjecture generation processes that can lead students to produce a proof by contradiction (Antonini, 2003b), to propose

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innovative didactic approaches to indirect proofs (Polya, 1967, pp. 163-171). Similarly differences were described in relation to how verbal and symbolic context affect students' performances when dealing with contraposition equivalence (Stylianides et al., 2004). The aim of this paper is outlining a frame allowing a uniform and systematic approach overcoming the fragmentary of previous contributions. What we present is an interpreting model of indirect proof, as well some examples aimed to illustrate its effectiveness.

METHODOLOGY

Our project developed through two main research directions, empirical and theoretical; the dialectical relationship between them contributed to the construction of the interpreting model. Empirical data were both qualitative and quantitative and were collected according to different methodologies: individual interviews, written tests, observation and recording of classroom activities, mainly collective discussion. Empirical investigations concerned both high school and university students (in particular students of scientific faculties: Mathematics, Physics, Biology, Pharmacy).

THE MODEL

The elaboration of the model started from the analysis of indirect proof that we framed within the model introduced in (Mariotti et al., 1997; Mariotti, 2000) through the ‘didactic’ definition of ‘mathematical theorem’. According to such a definition, a mathematical theorem is characterized by the system of relations between a statement, its proof, and the theory within which the proof makes sense. In the following the tern constituted by Statement, Proof and Theory, will be referred as (S, P, T).

The refinement of this definition was elaborated with the objective of taking into account two basic aspects: the logical structure of an indirect proof and the distinction between Theory and Meta-Theory.

Direct and indirect Proof

First of all, let us try to clarify what we mean with the expression *indirect proof*. In fact, the use of the expressions "indirect proof", "proof by contradiction", "proof by contraposition", "proof ad absurdum" in the textbooks is far from being clear and uniform, while its use may be considered controversial even among the mathematicians (Bernardi, 2002; Antonini, 2003a). In general, these expressions denote one or both of the types of proof which, in this paper, we call *proof by contradiction* and *proof by contraposition*. From logical point of view, there are important differences but also several relationships between them that we do not treat here; in the teaching of mathematics, in Italy, both of them are generally named “*dimostrazione per assurdo*” while sometimes this expression is used only for proof by contradiction (Antonini, 2003a).

In this paper, by *indirect proof* we refer to both *proof by contradiction* and *proof by contraposition*. In other words, consider a statement S and assume that it can be formulated as an implication, $p \rightarrow q$; let us name *direct* the proof P if among the statements that constitute the deductive chain does not appear the negation of the thesis p. If this is not the case, we will speak of *indirect proof*. Elaborating on this basic definition we are going to describe the structure of indirect proofs. Consider the following examples.

Example 1

Statement: Let n be a natural number. If n^2 is even then n is even.

Proof: Assume n a natural odd number, then there exists a natural number k such that n=2k+1. As a consequence $n^2 = (2k+1)^2 = 4k^2+4k+1 = 2 \cdot (2k^2+2k)+1$, that means that n^2 is an odd number.

This is an example of "proof by contraposition". If the statement is expressed by the implication $p \rightarrow q$, the given proof is a **direct proof** of the new statement "*if n is odd then n^2 is odd*", that is the contraposition ($\neg q \rightarrow \neg p$) of the original statement $p \rightarrow q$.

Example 2

Statement: Let a and b be two real numbers. If ab=0 then a=0 or b=0.

Proof: Assume by contradiction that ab=0 and that $a \neq 0$ and $b \neq 0$. Since $a \neq 0$ and $b \neq 0$ one can divide both sides of the equality ab=0 by a and by b, obtaining 1=0.

This is an example of a "proof by contradiction", where is given a **direct proof** of the statement "*let a and b be two real numbers; if ab=0 and $a^{1}0$ and $b^{1}0$ then 1=0* ". The hypothesis of this new statement is the negation of the original statement and the thesis is a false proposition ("1=0").

In both examples, in order to prove a statement S, that we will call the *principal statement*, a direct proof is given of a new statement S*, that we will call the *secondary statement*.

Principal statement S	**Secondary statement S***
Let n be a natural number. If n^2 is even then n is even.	Let n be a natural number. If n is odd then n^2 is odd.
Let a and b be two real numbers. If ab=0 then a=0 or b=0.	Let a and b be two real numbers. If ab=0 and $a \neq 0$ and $b \neq 0$ then 1=0.

Table 1

Principal statement and secondary statement in two indirect proofs

The two examples of proof share a common feature in the shift from one statement (principal statement) to another (secondary statement) and base the acceptability of this shift on particular logical relationships valid in the meta-theory.

In both cases, it is possible to prove that an indirect proof of the principal statement can be considered accomplished if the *meta-statement* S*→S is valid; in fact, in this case, from S* and S*→S it is possible to derive the validity of S by the well known "modus ponens" inference rule. But, the validity of the implication S*→S depends on the logic theory, i.e. the *meta-theory*, within which the assumed inference rules are stated. As it is commonly the case, i.e. in the *classic* logic theory, such a *meta-theorem* is valid, but it does not happen in other logic theories, such as the *minimal* or *the intuitionistic* logic[1].

Of course, no trace of this meta-theoretical elaboration is made explicit in the proofs as they are usually presented both the textbooks and in the courses.

A model of indirect proof

According to the previous analysis three key statements and three key theorems are involved in an indirect proof. The statements are: the principal statement S, the secondary statement S* and the implication S*→S, that we can name *meta-statement*, referring to its meta-theoretical status. The theorems are:

1. the *sub-theorem* (S*,C,T) consisting of the *statement S** and *direct proof* C based on a specific mathematical theory T (Algebra, Euclidean Geometry, and the like);
2. a *meta-theorem* (MS, MP, MT), consisting of a *meta-statement* MS=S*→S and a *meta-proof* MP based on a specific *Meta-Theory*, MT (that usually coincides with classic logic);
3. the *principal theorem*, consisting of the *statement* S, the *indirect proof of* S, based on a theoretical system consisting of both the *theory* T and the *meta-theory* MT.

Let name an *indirect proof of S a pair consisting in the sub-theorem (S*,C,T) and the meta-theorem (MS,MP,MT); in symbols P=[(S*,C,T),(MS,MP,MT)]*. In summary, an indirect proof consists of a couple of theorems belonging to two different logical levels, the level of the mathematical theory and the level of the logic theory.

Statements	Proofs	Theoretical levels
S*	C direct	Theory T
S*→S	MP	Meta-Theory (MT)
S	(S*,C,T)+(MS,MP,MT) indirect	T+MT Theory and Meta-Theory

Table 2
Analysis of an indirect proof. We highlighted the only elements usually made explicit in a mathematical proof as we read it in a textbook.

[1] For a definition in terms of rules of inference of the classic, minimal and intuitionistic logic, see Prawitz (1971).

HYPOTHESES EMERGING FROM THE INTERPRETING MODEL

The model presented above emerged from an a priori structural analysis but can be reinvested to describe and analyse students' cognitive processes, involved both in producing and interpreting indirect proofs. In particular, this model allows to identify key elements to formulate research hypotheses concerning the potential source of students' difficulties. In the following we will discuss some of them.

1) The shift from the proof of the principal statement to the proof of the secondary statement may present specific difficulties; in fact, the relationship between the two proofs may not be so *intuitively acceptable* (in the sense of Fischbein, 1987) as is commonly assumed.
2) Cognitive conflict may be expected to emerge in either producing or interpreting the proof of the secondary statement, when inferences are made on the base of openly false hypotheses.

We are going to illustrate these hypotheses through the discussion of two examples; we hope also to illustrate the effectiveness of the model in framing the analysis. In the transcript of the interviews the letter "I" indicates the interviewer while the initial of the name indicates the student. Emphasis is expressed by bold characters.

Difficulties in the shift from the principal to the secondary statement

Fabio is a university student (last year of the degree in Physics), he was asked to express his opinion on the indirect proof [2].

PROTOCOL	ANALYSIS
F: Proof by contradiction is artificial: how does one get out of it? Ok, you have arrived to the contradiction… and then? […] I don't see that conclusion be linked to the other one, I miss the spark […]	Fabio clearly express his difficulty to grasp the link between the two statements S and S*: according to our model the source of difficulty seems to be the meta-theorem.
I: Let's make an example: we take a natural number n. Theorem: if n^2 is even then n is even. Proof: if n is odd I write n=2k+1, then... [the interviewer writes down algebraic transformations] $n^2=2(2k^2+2k)+1$ is odd.	The interviewer proposes an example. The principal statement is: S : *if n^2 is even then n is even.* The indirect proof consists of the direct proof of the secondary statement, S*, that remains

[2] Fabio and the interviewer use the expression "proof by contradiction" to denote "proof by contraposition". As previously reported, many students and teachers in Italy, name in this way both proof by contradiction and proof by contraposition, without distinguish between the two types.

	unspoken: *if n is odd then n^2 is odd.*
F: Yes, I understand, it is better to prove that if n is odd then n^2 is odd.	Fabio acknowledges the advantage of the shift to the secondary statement S*, that he makes explicit.
I: And then, which is the problem?	
F: The problem is that in this way we proved that n is odd implies n^2 is odd, and I accept this; but I do not feel satisfied with the other one.	Fabio clearly express his feeling. He can identify the two statements, he is ready to accept the given proof, C, as a proof of S* ("*we proved that n is odd implies n^2 is odd, and I accept this*") but not as a proof of S ("*I do not feel satisfied with the other one*").
I: Do you agree that natural numbers are odd or even and there are not other possibilities?	
F: Yes, of course... and now you will say: n^2 is even, n is even or odd, but if it were odd, n^2 would be odd, but it was even... yes, ok, I know, but…something escapes me.	Fabio shows that is able to produce an argument to explain the method of the indirect proof, nevertheless there is something that he is not able to grasp ("*something escapes me*").
F: First of all, why do I have to begin from n not even? I don't see immediate conclusion. And, at the end: "then it can not be other than n even", it is a gap, the gap of the conclusion... it's an act of faith... yes, at the end it's an act of faith.	The move from the proof of the secondary statement to the validation of principal statement is not immediate, on the contrary is not rationally acceptable.

As the analysis of the protocol shows, the acceptability of the proof of the statement S* does not immediately entail that the principal statement was proved. The feeling of distress (upset) openly expressed by Fabio was also observed in other studies. For instance Stylianides et al. (2004) observed a similar resistence:

"*some students reject the contraposition equivalence rule because they believe that the correct equivalence relating the conditional statement $p \Rightarrow q$ with the proposition $\neg p$ and $\neg q$ is $p \Rightarrow q \equiv \neg p \Rightarrow \neg q$*" (p. 149).

What makes this protocol so peculiar is the fact that the introspection ability of Fabio let us know where the conflict arises. According to our hypothesis and with the terminology of our model, the difficulty can be localized in the meta-theorem, that is in the cognitive difficulty of grasping as immediate the logical link expressed by the implication S*→S. For Fabio, and probably for many other students, such a link is not an intuition (in the sense of Fischbein, 1987; for other details on intuition and the shift between statements, see also Antonini, 2004) and accepting it is cause of distress (“*I don’t see immediate conclusion*”, “*I do not feel satisfied*”, “*something escapes me* ”).

Difficulties in identifying the Theory of reference

The following example concerns specific difficulties related to producing a proof of a secondary statement, when the inferences should be based on openly false assumptions. Maria is a university student of the last year of Pharmacy. (For other details on the proof of the secondary statement, see Antonini & Mariotti, 2006).

PROTOCOL	ANALYSIS
I: I: Could you try to prove by contradiction the following: “if ab=0 then a=0 or b=0”?	The interviewer asks Maria a proof by contradiction.
M: [...] well, assume that ab=0 with a different from 0 and b different from 0... I can divide by b... ab/b=0/b... that is a=0. I do not know whether this is a proof, because **there might be many things that I haven’t seen**.	Maria produces such a proof, but she is doubtful about its validity.
M: Moreover, so as ab=0 with a different from 0 and b different from 0, that is against my common beliefs [in Italian: “*contro le mie normali vedute*”] and I must pretend to be true, **I do not know if I can consider that 0/b=0. I mean, I do not know what is true and what I pretend it is true**.	Difficulties emerge about the validity of the sub-theorem: upsetting fundamental beliefs seems to be the cause. Maria declares that she lost the control on what is true and what is false.
I: Let us say that one can use that 0/b=0.	
M: It comes that a=0 and consequently … we are back to reality. Then it is proved because … also in the absurd world it may come a true thing: thus I cannot stay in the absurd world. **The absurd world has its own rules, which are absurd,** and if one does not respect them, comes back.	Maria considers “*absurd*” the “*world*” where the false hypothesis of the secondary statement is assumed. The “*rules*” (Theory) in respect to which the proof of the sub-theorem makes sense, belong to an "*absurd world*"; these rules are absurd too, they may not coincide with the rules commonly applied.
I: Who does come back?	Where we accept something false, Maria claims, whatever can happen, included 0/b≠0. The absurdity of the hypothesis of the secondary statement conflicts with use of the 'common' Theory; Maria thinks that T should be
M: It is as if a, b and ab move from the real world to the absurd world, but the rules do not function on them, consequently they	

have to come back …

M: But my problem is to understand which are the rules in the absurd world, are they the rules of the absurd world or those of the real world? **This is the reason why I have problems to know if 0/b=0, I do not know whether it is true in the absurd world**. […]

replaced by a new theory T*, different and more adequate to the *absurd world* where the proof is deplaced. The *real world* has its own rules, different from those of the *absurd world* .

I: [The interviewer shows the proof by contradiction of "$\sqrt{2}$ is irrational", than asks] what do you think about it?

M: in this case, I have no doubts, but why is it so? … perhaps, when I have accepted that the square root of 2 is a fraction I continued to stay in **my world**, I made the calculations **as I usually do**, I did not put myself problems like "in this world, a prime number is no more a prime number" or "a number is no more represented by the product of prime numbers". The difference between this case and the case of the zero-product is in the fact that this is obvious whilst I can believe that the square root of 2 is a fraction, **I can believe that it is true and I can go on as if it were true**. In the case of the zero-product I cannot pretend that it is true, **I cannot tell myself such a lie and believe it too!**

Differently from the previous case, Maria declares that she is comfortable with this proof; in fact, assuming the $\sqrt{2}$ is rational does not present any difficulty for her: the fact that $\sqrt{2}$ is rational is plausible. Consequently, in the *absurd world*, where $\sqrt{2}$ is rational, the basic truths are not questioned ("*I can believe that it is true and I can go on as if it were true*"), the Theory of reference is not upset.

The source of Maria's difficulties can be found in the difficulty of managing the upset of well settled rules caused by an assumption that is so evidently false for Maria that she cannot even pretend that it is true without upsetting the whole theoretical frame within which any proof can make sense. According to our model, Maria difficulties concern the sub-theorem (S*,C,T) and in particular the identification of theory to which the proof C refers. Maria hypothesises that a new theory T* should be used, different but more adequate to the absurd situation generated by the false assumption, in which she does not know whether 0/b is equal or not to 0.

This result agrees with what expressed by Durand-Guerrier (2003): implication with false antecedent are not accepted or however are considered as false by students. In order to accept a proof it seems necessary to start from true, or at least potentially true, assumptions.

CONCLUSIONS

Although the brevity of this text does not allow further discussion, we hope that the previous examples gave an idea of how the interpretative model may function in the analysis of students difficulties. In particular the model allows to clarify the articulation of the different theorems involved (*principal theorem, sub-theorem* and *meta-theorem*), and the articulation of the different theoretical levels (*theory* and *meta-theory*). A generic difficulty related to indirect proof can be consequently analysed in a more refined way, focussing on different aspects, showing the appearance of quite different kind of 'difficulties'.

An open research question is how to design teaching interventions aimed either to foster students' introduction to indirect proofs or to help students to overcome cognitive conflicts. We think that the model could show its effectiveness for planning didactic interventions, but in this regards, further investigation is necessary.

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