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Epistemological Problems with the Limit Concept – a case study On Communication and Argumentation within a computer-based learning environment

Abstract: Based on a paradigmatic case study of a classroom situation in which two students deal with the limit concept within a computer-aided learning environment, problems of concept development which are based on epistemological obstacles are investigated. The analysis of these problems stresses the importance of a genetic concept development especially in calculus lessons. At the same time it will be advocated for the need and the importance of descriptive working methods in research of advanced mathematical thinking.

1 Calculus lessons and qualitative teaching research

Within the last two decades qualitative research methods have become increasingly established even in the field of mathematics education, and are today accepted to a great extent in addition to conventional quantitative methods. Their operational field lies where research interest is directed to an area that can *not* be described by directly readable metric data. This is particularly the case if the task is to reconstruct, as detailed as possible, the learner's individual strategies and subjective ideas (compare Mayrink [1985]), or to document, to grasp and to explain processes of individual conceptual development.

For the purpose of teaching and learning in all school stages questions that deal with the individual conceptual development are in the center of interest, even and especially for calculus lessons. Nevertheless qualitative methods have hardly penetrated this area, on the whole their representatives have been working thus far on elementary questions of primary school and the intermediate stage[1].

In presenting the following case study I would like to plead for an increasing use of prescriptive methods even in calculus lessons. For this purpose I chose a classroom scene in which the students deal with one of the central concepts of calculus. It concerns problems with the *limit concept* and illustrates central questions of calculus lessons:

- How important is a *genetic development* of the limit concept, taking historical preliminary stages into account?

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[1] Compare e.g. Krummheuer [1985], Maier & Voigt [1991], Hölzl [1994], exceptions are Steinbring [1990] and Warmuth [1995]

- Which role does the clearing of *epistemological obstacles* - in the sense of Brousseau [1983] and Sierpinska [1992] - play?
- Is it necessary for students to *independently* think through the factual problems that are linked with the limit concept, or is a clever didactic processing able to relieve the students of this effort?

From a descriptive point of view the results of the following case study will back up to a great extent Sierpinska's [1992] theories about the importance of the clearing of *epistemological obstacles* for the learning of mathematics.

2 Software and teaching context

The case study is based on the documentation of a teaching project which was realized for several months in a regular calculus beginner's course.

In this classroom project half of the time was used at the students' disposal for problem-orientated working phases which were carried out in small groups in a computer-based learning environment. Throughout this situation the students used the interactive computer algebra-system (CAS) called *MathView*.

Already at the beginning of this set of teaching units serious deficits in the understanding of limit concept came up in a repeating phase. Even though these units had been taught one year ago students had problems in differing between the quotient of the differences $\frac{f(x+h)-f(x)}{h}$ and the derivative $f'(x)$. Almost completely unknown was even any kind of conceptual interpretation or use meaning[2] of these notions.

Therefore a training phase was inserted in order to illustrate the term $\frac{f(x+h)-f(x)}{h}$ once again, this time using the example of an exponential function. In order to make transparent the meaning of a variation of the difference h in algebraic as well as in graphic terms the computer algebra-system was used. For this purpose the following task was posed (see figure 1):

> Create a graph of function: $y_1 = 2^x$.
>
> Let's take a look at the term $m = \frac{f(x+h)-f(x)}{h}$ for the function $f(x) = y_1 = 2^x$ and let be $x = 0$. Enter the term and choose an initial assignment for h, e.g. $h = 2$.
>
> Now take a look at the straight line that goes through point (0;1) and which has gradient m.

[2] In the sense of Usiskin [1991]

Create a graph of this straight line in the same graphics window.

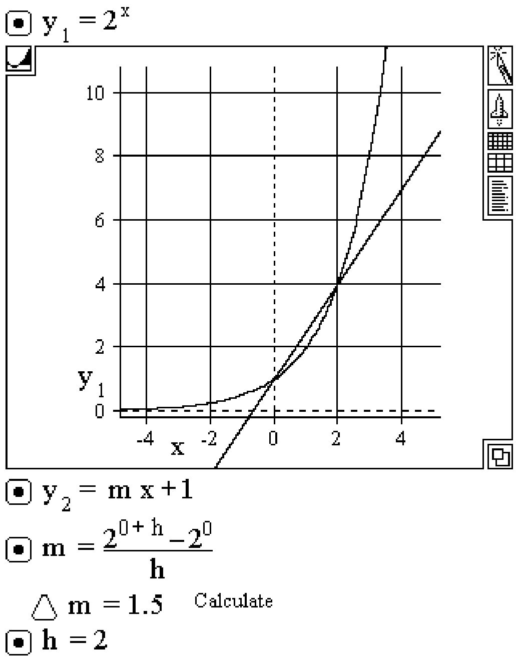

Figure 1: MathView screen graph, h = 2

Helped by the interactive connection, which is realized by CAS MathView, h is linked with the secant - over m and y_2. If one changes the value of h (by overwriting), m will be accordingly calculated again, the secant will then move accordingly around the fixed point (0;1). In addition the following questions were asked:

Enter some numerical values for h and observe the effects thereof.

a) How does the changing of h influence the other terms, i.e. the graph?

b) Which straight line will be created if the value of h is 2? How steep is gradient m then?

c) Function $y_1 = 2^x$ illustrates the expansion of a bacterial colony, with a present surface doubling its size (in cm^2) every day. What does $h = 2$ in this case mean and which meaning does the according value have to m?

During the working phase all pupils answered these questions in writing. Their interpretation gives evidence for a productive role of the computer. Some pupils may well have come to grasp the connection of the structure of the term $\frac{f(x+h)-f(x)}{h}$ and the according graphical presentation for the first time. Finally the last question on the worksheet deals with the starting point of the now following transcript:

e) Find an approximative solution for the tangent gradient at the point (0;1) as precisely as possible. What does this value mean if one comprehends function $y_1 = 2^x$ as a growth process as in question c)?

Let's observe - in the following scenes two students, Anka and Julia, tackling this task.

First scene

Julia: What does this mean? I have no idea what limit...
Anka: Limit means limit point. And if h moves towards zero, that means if h gets smaller...
Julia: Like we did it right now, one millionth?
Anka: Then we will get the limit, if you type that in like this.
Julia: But what actually is the limit?
Anka: This is the limit. *Points at m*
Julia: m!?
Anka: Yes.
Julia: The limit to what?
Anka: The limit to the tangent,... actually, from secant to tangent.
Julia: What exactly is the limit now?
Anka: It will always remain a secant. It is just an approach to the tangent. The limit is a most... You know, that is, right, we had that before. Okay...

Already in this small passage two different basic attitudes can be noticed which influence the students' thinking:

- Julia, as the questioning person, would like to know what the *limit* is.
- Anka on the other hand explains in her comments mainly how to *get* to the limit. She tries to answer Julia's question about the limit by describing the *limit process*, at the same time her explanations are not always free of contradiction and partly ambivalent.

Behind this object-process-differentiation there is a more serious problem to be seen: The connection between the links of a convergent sequence and the sequence limit, and thus the *relationship between intuitive basis of comprehension and formal-mathematical specification*. The factual basis of this strained relationship can be illustrated by the following tabulated comparison:

	intuitive comprehensive basis	*formal-math. specification*
sequence	lining up of elements, as a rule according to a regularity	function: $N \longrightarrow \mathfrak{R}$
sequence convergence	aiming for an objective, condensing more and more	convergence criteria
sequence limit	the aim of the process, which maybe will never be reached, yet is thought of	definition of the limit

	as an ideal completion; it symbolizes the locality of the thickest condensing	

To the learner of mathematics a difficult intellectual challenge lies in the following facts: As a rule the limit is not part of the sequence, but rather the ideal supplement (the existence of which is guaranteed by the completeness of the real numbers). This new element proves to be „ideal“ in two senses: On one hand it satisfies the intuitive requirements mentioned above to an optimum, on the other hand on a formal plane it can be described as a theoretically well determined object.

The process of the approximative determination of limit points - e.g. by calculus of sequence links with a place numeral which is high enough - is based on the optimal adaptation of the limit to the development process of the sequence. This can provide the misconception of the idea to „get to“ the limit once and for all in this way. However, a theoretically precise determination is possible only by refraining from the examination of separate sequence links.

In Anka’s argumentation some of these aspects and problems become obvious: She overlooks the development of the sequence (13) and the possibility to determine the limit approximately (16). Furthermore she knows that the limit which has to be calculated in this present case „belongs to the tangent“ and thus „in fact“ concerns the transition „from secant to tangent“ (23-24). At least she seems to know in addition that the approximative process does not „actually“ create the tangent, as it will „always remain a secant“ (26). It is not clear at this point how far, or if, Anka has understood the difference between the approximative and the precise limit determination. We will have a closer look at these connections later on.

We take a look at another scene. One after the other Anka and Julia assigned the values for h: 1/10, 1/100 and finally 1/1000 (see figure 2). Their conversation now concentrates on the question: what will happen with gradient m, i.e. with the secant, if h becomes smaller and smaller. Will m become „smaller and smaller“ as well, so that the corresponding secants will approach a horizontal? Julia expresses her presumption of such a connection, Anka argues against it.

Second scene

Anka: Well, we have still got these two points that move towards each other.
Julia: But if we make h into zero, then there won’t be any point any more, will there? You see, there won’t be any difference left...
Anka: The gradient, we’re only talking about the...
Julia: No, we’re talking about h. But m is the gradient, h is not the gradient.
Anka: That’s right, but we make this so small that

the distance will become so small, and this
can not *She holds a pen in front of the screen*
to create a horizontal. You see, this will
never become zero. The gradient will never
become zero, or otherwise it would cut right
through there. This always has to be a little
bit diagonal next to it.
Julia: But why? I really don't understand it.
Anka: *Points with the mouse at the secant*. This
thingy over here, this thingy here, that can
never lay straight.
Julia: Why?
Anka: Because it still has to keep the outer contact
with this one minipoint over there!

Already in the first scene different basic attitudes were presented in the interpretation of what "limit converging towards 0" means. They can be found also at the beginning of this scene. Yet the main problem the students are fighting with is situated on a more intense plane: Their thoughts circle around a basic insight for which mathematicians have been struggling throughout the history for a long time. Its articulation sounds as follows:

- *Geometrically*: Two points moving towards each other can determine a well determined limit direction, i.e. a corresponding limit straight line.
- *Arithmetically*: A quotient sequence with denominators converging towards zero can converge towards a well determined finite value.

With a pen that Anka holds in front of the screen graph, she illustrates how a straight line with the gradient zero would go through the examined point, and explains: "You see, this can never become zero" (106-108). The thought of two converging points determines her intuition. The "thingy" - meaning the secant - can never lay straight, "because it still has to keep the outer contact with that minipoint over there" (111-116).

What is a *minipoint* to Anka? There are signs in Anka's argumentation for the assumption that in her idea of the process of h becoming "smaller and smaller" there is a kind of *in between world* of the infinite small, namely when h is smaller than any imaginable number, but still bigger than zero. The "minipoint" seems to be an element out of this world of the "infinite small". Points can normally be marked, yet the minipoint can not be grasped. Nevertheless it has got an important function: It stabilizes the secant in the sense of an outer contact and thus makes the limit position of the secant, that means the tangent, calculable. At the same time the attachment to the minipoint prevents the degeneration of the tangent, that it e.g., in form of a horizontal, "cuts entirely through here" (106-107).

Anka's minipoint-argumentation is not understandable for Julia. Her comprehension requires the idea of a *process* of two converging points. Julia on the other hand thinks about <u>one</u> point and pursues her thought:

Third scene

136 Julia: But if I make h into zero...
137 Anka: Well, if I make h into zero, that's completely
138 different then.
139 Julia: Well, that's not possible then. You're not
140 allowed to make h into zero.
141 ...
142 Julia: You're not allowed to make h into zero,
143 because right down over there you can...
144 *points at the quotient on the computer screen*
145 *...*well, I think it's therefore that I'm not
146 allowed to divide by zero.

What do the two students mean when they say "make h into zero"? Do they want to express that h is supposed to approach zero more and more or that one should give h the value zero? A lot is in favor of the second interpretation: For a long time Julia has been showing a tendency to connect the object "tangent" with the insertion $h = 0$. Up to now Anka has consistently stuck with her process-idea. Now she seems to carry out a change in the dominant explanation model: Up to this point her argumentation has given the impression that she understands the tangent as a limit position of the secant. Later on she possibly assumes that the tangent may have an existence beyond this limit process, namely that h does not move towards zero, but rather that h *is* zero. This being the case, she believes that something "completely different" (137) may happen, but what?

Julia sees a problem if one "makes h into zero", and refers to the fact that one can't divide by zero (142-146). Anka does not contradict and nods in approval. The more astonishing is the further course of events:

Fourth scene

150 Anka: *Moves cursor to 'declarations'*. Now, let's
151 close this window. So, and now let's make h
152 into 0... *Inserts 0 for h and the secant*
disappears
153 Julia: What is happening now?
154 Anka: Well, it doesn't exist any more!
155 Julia: Why?
156 Anka: Because...

Julia: You said it would be straight if h were zero!
Anka: I did not say that! I didn't even know it myself!
That's right, and as this... right, that's not there any more. Now we even haven't got a distance between the two points left. There we still had a minidistance, and so it could go right through it, couldn't it, at these two points. And as h is zero now, there's only x left.

Now the computer is used in order to get closer to the mysteries of the limit - but the experiment proves to be a failure. Anka enters the assignment zero for h and the students eagerly wait for the result. Disappointed they discover that for the insertion $h = 0$ the computer does not create a graph. On the screen the previously shown secant disappears. Anka admits that she did not know herself what would happen.

But soon she manages to overcome her helplessness. She returns to her previous conceptual model and soon understands why the computer can not produce a secant for $h = 0$: "There we still had a minidistance, and so it could go right through it, couldn't it, at these two points. And as h is now zero, there's only x left." (162-166). Anka saved herself by entering into an intellectual area that provides her with safety: Her process understanding is able to explain the reaction of the computer.

The *minidistance* may be - like the minipoint - an obscure element of the *in between world of the infinite small*. It is interesting that for Anka the idea of the minidistance nevertheless contains a supporting function: One is not able to directly see or perceive it. Yet its existence is well shown in the secant which can only exist if there is at least a minidistance. If the distance becomes zero, the entire construction will collapse.

3 Summary of examination and conclusion

The fullness of higher problems and phenomena, which are based on epistemological aspects of the mathematical contents, occurring in these short transcript extracts is amazing. They circle around *the intuition of the infinity and their mathematical specification*. Main problem fields occur in the contrast respectively in the relationships between

- graphic and arithmetical representation of mathematical content,
- process and object,
- static and dynamic interpretation,
- intuitive idea and mathematical specification.

In normal lessons there is often not enough scope for the active attempt to come to terms with these kind of topics. More often the attitude is present that a reflecting attempt at dealing with problems of this kind does not play any role in basic course students' thought. On the other hand the dialog of this case study gives a different impression. Some students' thoughts remind of early comments of famous mathematicians during the original phase of infinitesimal calculus; e.g. the characterization of the differential quotient by Leibniz:

> "*Dx* does not approach zero. It is more likely the case that the 'final value' of *Dx* is not 0, but an 'infinite small value', a 'differential', called *dx*, and thus *Dy* has a 'final' infinite small value *dy*. The quotient of this infinite small differential is finally a common number again, $f'(x) = dy/dx$" (Courant & Robbins [1973], p. 330)

The *minipoint* and the *minidistance* in Anka's thinking could well correspond to these "final" infinite small values. In this example it becomes very clear that the confrontation with the infinitesimal gives new experiences concerning the dealing with mathematics. This is connected with changes concerning the intuitive level and requires the generation of new basic ideas (compare vom Hofe [1998a]).

The *computer-aided learning environment* plays a fundamental role: it plays a major part in the making of these scenes, in fact this new attempt to arrive at a conceptual clarification of the limit concept is much provided by the special learning environment. The computer may not be able to solve the substantial problems; yet it is by this inability that it helps to discover their true nature. The computer is of central significance in the development of communication: it creates a common field of experience in which students can discover a new contact with mathematics.

I would like to conclude by presenting three short theses; they concern the areas of (1) mathematical concept development, (2) the use of computers in mathematics lessons and (3) case studies as a research method and thus reflect three different levels of this paper:

(1) *Genetic concept development and epistemological obstacles*. Learning in the sense of the genetic principle requires provocation, processing and clearing of epistemological obstacles. It still remains an important responsibility of mathematics education to develop and to test learning and exercise forms for this purpose.

(2) *Interactive analysis-software*. Interactive analysis-software can provide even the inner mathematical concept development - apart from various application topics. Yet the possibilities of visualization linked with the software are not able to replace the power of productive ideas.

(3) *Interpretative case studies and analysis lessons*. Interpretative case studies are a worthwhile means of analyzing and understanding even for calculus lessons. They may contribute to the improving of our knowledge of students' thought strategies. With the liveliness and authenticity of descriptive studies we are able to complete, extend and enrich the didactics of calculus, which is usually characterized by prescriptive efforts.

References:

Brousseau, G. [1983]: Les obstacles épistémologique et les problèmes en mathématiques. In: Revue Recherches en Didactique des mathématiques, 4 (2), p. 165 - 198

Courant, R. & H. Robbins [1973]: Was ist Mathematik? Berlin, Heidelberg, New York: Springer

Hefendehl-Hebeker [1998]: Mathematik erlebnen zwischen Faszination und Fremdheit. In: *mathematik lehren*, 86, p. 4 - 7

Hölzl, R. [1994]: Im Zugmodus der Cabri-Geometrie: Interaktionsstudien und Analysen zum Mathematikunterricht mit dem Computer; Weinheim: Deutscher Studien Verlag

Krummheuer, G. [1985]: Rahmenanalyse zum Unterricht einer Klasse über "Termumformungen". In: Bauersfeld, H. u. a.: Analysen zum Unterrichtshandeln. IDM Bd. 5; Köln: Aulis, p. 41 - 103

Maier, H. & J. Voigt, Hrsg. [1994]: Verstehen und Verständigung. Arbeiten zur interpretativen Unterrichtsforschung. IDM Bd. 19 Maier, H. & J. Voigt, (eds.) [1991]: Interpretative Unterrichtsforschung; Köln: Aulis

Mayrink, P. [1985]: Qualitative Inhaltsanalyse. In: G. Jüttemann (Hrsg.), Qualitative Forschung in der Psychologie; Weinheim: Belz

Sierpinska, A. [1992]: On understanding the notion of function. In: Harel, G. & E. Dubinsky, Hrsg.: The concept of function - Aspects of epistemology and pedagogy; MAA Notes Vol. 25, p. 25 - 58

Steinbring, H. [1990]: Probleme der Entwicklung mathematischen Wissens im Unterricht - an einer Analysis-Stunde betrachtet. In: Der Mathematikunterricht, Jg. 36, Heft 3, p. 4 - 28

Hofe, R. vom [1998a]: On the generation of basic ideas and individual images: Normative, descriptive and constructive aspects. In: Kilpatrick, J. & A. Sierpinska (Eds.): Mathematics education as a research domain: A search for identity. Kluver Academic Publishers, p. 317 - 331

Hofe, R. vom [1998b]: Probleme mit dem Grenzwert - Genetische Begriffsbildung und geistige Hindernisse - Eine Fallstudie aus dem computergestützten Analysisunterricht. In: *Journal für Mathematik-Didaktik*, to appear 1998, 35 p.

Usiskin, Z. [1991]: Building Mathematics Curricula with Applications and Modelling. In: Niss/Blum/Huntley (Hrsg.) [1991]: Teaching of Mathematical Modelling and Applications; Chichester: Horwood, 30 - 45

Warmuth, T. [1995]: Untersuchungen zum Einsatz von Computeralgebrasystemen beim Bearbeiten realitätsorientierter Aufgaben im Analysisunterricht. Dissertation: Universität Kassel

Wittmann, E. Ch. [1991]: Mathematikunterricht zwischen Skylla und Charybdies. In: Mittteilungen der mathematischen Gesellschaft in Hamburg; Band XII, Heft 3; Festschrift zum 300-jährigen Bestehen der Gesellschaft; Dritter Teil; p. 663 - 679

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