THE PLACE AND SIGNIFICANCE OF THE PROBLEMS FOR PROOF IN LEARNING MATHEMATICS

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The paper deals with both theoretical questions connected with the nature of proof, and practical questions on teaching mathematical proofs to pupils and students, various types of problems are presented from secondary school course as well as graduate course. The author provides typology of proofs in general taking to attention the connection of intuition with the original nature of the ideas put forward. The article stresses the importance of the history of mathematics in teaching mathematics, considers some classroom situations, an outline of typology of proofs for teaching to prove is given.

Introduction

At present mathematical methods are widely used in sciences and humanities. It contributes to the growth of interest towards the very essence of mathematical reasoning and the nature of proof. Formal theories of mathematical proof have been constructed (Manin, 1979); the classifications of proofs according to different criteria, such as method, style and form, have been considered (*Ibañes* & *Ortega*, 1996); as well as for various age groups (Miyazaki, 2000). Great attention has been paid to the role and function of proof (de Villiers, 1987, Hanna, 2000), teaching approaches of proof (Hoyles&Healy, 1999), the view about proof (Lerman et al., 1993). The present paper, not focusing on individual classifications, is aimed at the consideration of some particular questions of the nature of proof and the practical application of different problem types in learning and performing proofs by pupils and students. Because of the lack of space, the part devoted to the classroom situations has been considered briefly, and only the outline of possible studying situations has been made.

View on the nature of Proof

All the problems on proof ever faced by the man, may be divided into those already *proved* (they were proved by someone for the first time) and those *not proved* (for the present moment). Let us consider the following scheme (Figure 1.). A not proved problem when proved by someone, moves to the set of proved problems. Let us call such a proof *a pure proof*. The proofs of the proved problems which are of scientific value and are not used in teaching, will be called *scientific proofs*. For example, the proofs of two Burnside's hypotheses from finite group theory contain about five hundred pages each.

The proofs of the proved problems which are used in teaching mathematics will be called *teaching proofs*. In their turn, teaching proofs will be subdivided into those made by a pupil himself (*independent proof*) and the proofs explained by a teacher (*auxiliary proof*).

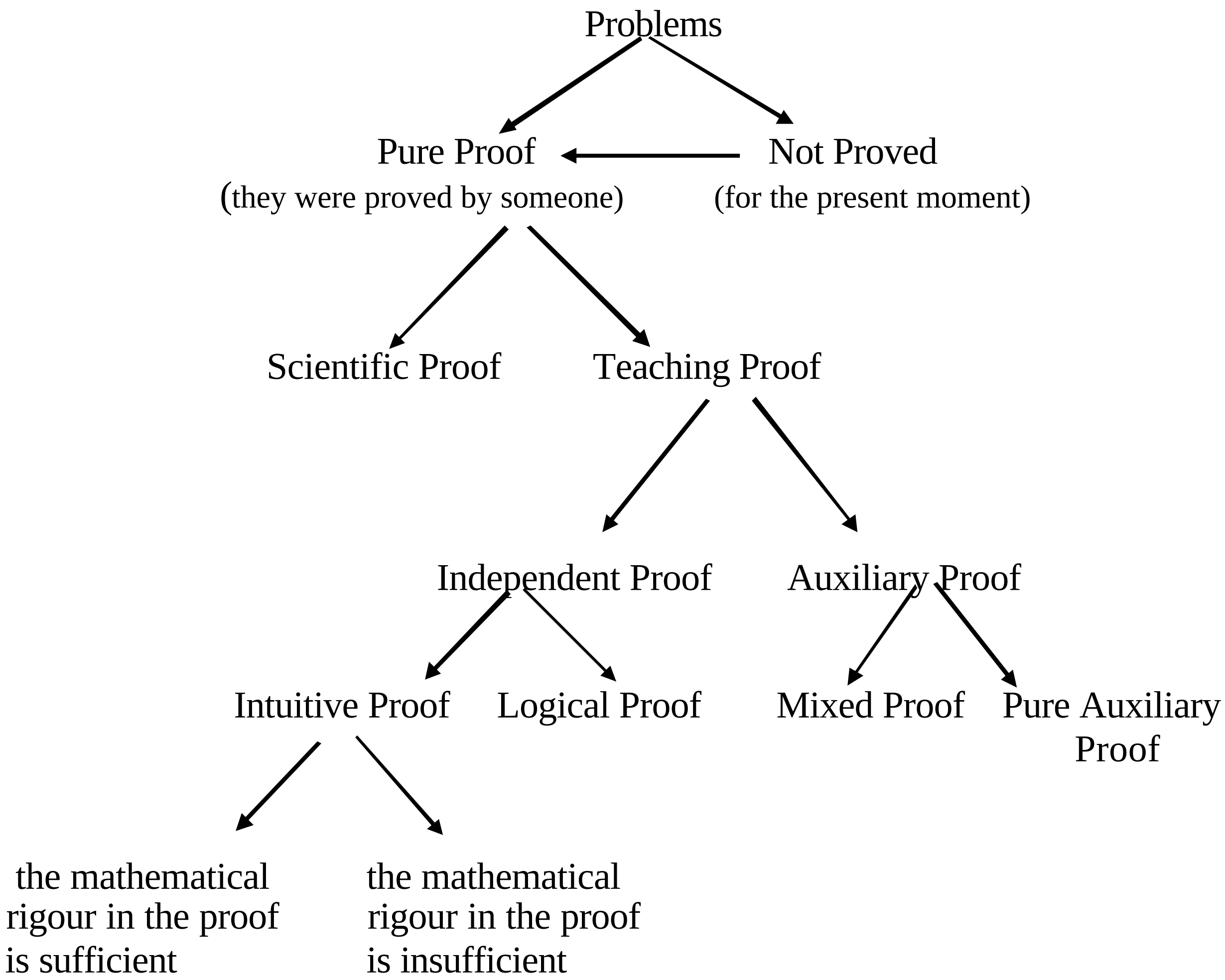

Figure 1. Typology of proofs in general

Intuition in mathematical proofs is inseparably connected with the originality of mathematical thinking, with creativity while proving. Modern usage of the term "*intuition*" originates from Descartes. Russian mathematician Steklov (1923) stated that "the method of discovery and invention is the same for all, one and the same *intuition* , because nobody discovers anything with the help of logic; a syllogism may lead other people to the agreement with that or other proof known before, but as a tool of invention it is useless… But the heart of the matter is that even in simple cases it is impossible to logically explain all the stages of proof. In invention of practically every step of proof it is intuition that matters and not logic; intuition is higher than any logic". Independent proofs, thus, can be divided into proofs where intuition is present (the so-called *intuitive proofs*), and the proofs which will be called *logical proofs*, i.e. proofs made only with the help of logic, in other words proofs where one uses the method known to a pupil and leading to a purpose though not demanding to put forward new ideas, while the proofs with the use of intuition are necessarily connected with the presence of originality in the ideas proposed by a pupil.

In order to consider the proof connected with the use of intuition more deeply, let us refer to Steklov's (1923) example. How to prove that in an isosceles triangle angles by the base are equal(Figure 2.). Euclid continues equal sides AB and CB and on their continuation marks equal line segments AD and CE; then point D is connected by a straight line with point C, and point E with point A. The investigation of the drawing acquired leads to the needed result at once. A listener is asking himself: why did Euclid wish

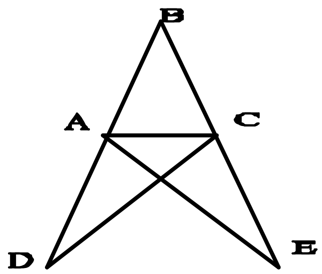

Figure 2.

to make such a construction? Later he will understand that the theorem is stated by the above device, but how and why did this device appear in mind?

This is the fact that essentially cannot be explained. The geometrician's actions were caused by the intuitive ability of invention that cannot be explained by logic, i.e. by a syllogism. One may logically state that if one acts in the way suggested by Euclid, the equality of angles BAC and BCA may be proved by a number of syllogisms, but the question "why did he make the very above construction?" may be accepted as a fact not allowing any explanations in itself; or the explanation is reduced to the fact that Euclid acted in the above manner because the earlier stated truth is proved in such a way.

The farther we will move from elementary things, the more evident this feature will become. The result is guessed in advance intuitively, often by the analogy with particular cases noticed by the few, and the proof devices are invented in the same manner. Further , let us divide intuitive proofs into *the proofs where the mathematical rigour is sufficient*, and *the proofs with insufficient mathematical rigour*. In our opinion, both types of proofs are extremely important here while teaching mathematics of both pupils and students. In this respect, the case of Euler proving the great Fermat theorem for $n=3$, is conspicuous. On the one hand, we have a classical example of an intuitive proof with insufficient mathematical rigour and, of course, an original idea will never substitute a full proof. But the former is as important as the latter; there will be no proof without an idea, therefore it is extremely important "to discover" in students the ability to put forward ideas and, as Polya (1954) puts it, to distinguish a more reasonable idea (guess) from less reasonable.

Now let us dwell on auxiliary proofs without which learning mathematics will not be complete. An auxiliary proof in which a key point of the proof is proposed by a pupil independently with the support of a teacher, will be called *a mixed proof*. And the proof where a key point has been explained by a teacher, as well as the whole proof, will be called *a pure auxiliary proof*. No doubt, the role of mixed proofs in teaching process is significant. It is very important that the key point while performing a proof even by a teacher, should be proposed and triggered by a pupil. The task of a teacher is to bring pupils up to this action, i.e. to provide all the prerequisites so that an idea will become “alive” in pupil’s mind in a conscious form. This should be paid attention to beginning with the earliest stages, namely, from primary school, every time using the material suited for the age.

This scheme of typology of proofs in general presupposes possibility of being considered from various points of view. From a teacher’s point of view, a teaching problem is only such a problem and may be classified according to some complementary indication (as in the part on various types of problems). From a pupil’s point of view, the same teaching problem is not proved *for him* , therefore the terms “pure proof” and “independent proof” in this case coincide. Also, not showing in the scheme, let us note that scientific proofs not having teaching purpose, will consist practically wholly of the proofs connected with the use of intuition with sufficient mathematical rigour.

Beginning with Ancient Greece and Egypt till today, every problem passes such a way (from the moment of its origin), and every pupil (student) proving the simplest teaching problem, passes through the stage of his own discovery. If one simplifies the situation considered and drops the dependence on many factors, one may state that the task of a pupil’s (student’s) mathematical teaching must be the most numerous passage of the phase of teaching problems. The question of the relation of independent and auxiliary proofs will be left opened, however let us point out that in each of the given types priorities should be made towards intuitive and mixed proofs. It is necessary to strive that the students, even on an elementary level, should have originality of the hypotheses (ideas) put forward, that is in its turn connected with intuition. Besides, not making a detailed comparison, let us point out that the above scheme is true both for the Graduate level and the Undergraduate level (up to the phase of teaching problems). The most talented pupils and then students became mathematicians and passed both the phase of the proved problems and that of the not proved ones, that is the whole scheme. It is logical to affirm that on the Graduate level, a student who has not received skills of independent thinking, developed his capacity of ideas proposition, i.e. covered the whole scheme up to the phase of teaching problems with priorities of proofs connected with the use of intuition and proofs, whose key points are suggested by a pupil, will experience serious difficulties while proving mathematical problems and significantly lag behind his group-mates who have received such a teaching training on the Undergraduate level.

Every problem and every proof to it, wherever it may find place in the given scheme, has its own historical value and significance which, unfortunately, is not often given proper attention nowadays. Proceeding from the above, one may speak of inseparable connection between each proof, each mathematical problem with the history of mathematics. Historical material provides additional basis for the understanding and mastering mathematical truths. In the former Soviet Union till the fifties of the 20th century, much more attention was paid to the connection of studying mathematical proofs with the problems of history of mathematics. In many respects, the new is the well forgotten old. As early back as in 1914, Kharkov mathematician D.M.Sintsov spoke on the permeating of teaching mathematics with the material on history of mathematics (Naumov, 1955, Yevdokimov, 2002). Materials on mathematical education stored and presented by many outstanding scientists should be used.

Any proof originally presents difficulty for pupils and students because it is necessary to apply one’s knowledge in a not clear order, unlike, for example, an algorithm of findings roots of a certain equation. Any proof problem should develop a pupil’s mathematical capabilities and contribute to his understanding mathematics as a whole. It is important to take into account the moment when pupils pass from a formal running over of possible ways (or, speaking more generally, known methods of proving), acting due to analogy or not consciously, to conscious considering of the actions performed on a given problem. It goes without saying that the proofs proving a certain statement from the point of view of calculation or logical proofs, for example, proving that $x_1,...,x_k$ are the roots of the equation $P(x)=0$, are needed. They constitute a significant part of mathematical education. However, every student getting mathematical education should be taught the ability to think mathematically, to analyze proposed actions and to study their possible realization. Having stated a certain statement on the function T_{W,h_0} mathematician Mumford (1967) wrote: ”This statement is made through very lengthy though quite simple computations. Having performed them in all details to the end, I spent several hours but did not become cleverer but I was just reaffirmed in the true nature of the definition. Therefore I will drop all the details here”. Summing up, we may say that a good proof is the consideration that makes us cleverer.

Using various types of proof problems in teaching

Not trying to embrace the whole range of proof problems necessary for a full mathematical training, let us concentrate on some types of problems which, in our opinion, are given very little time and significance while teaching mathematical disciplines both in secondary schools and universities (the material has been studied for the Eastern Ukraine). Let us note that the main thing is the format of a problem and not its theme, belonging either to school or university teaching course. The aim of this part is to give and characterize several types of problems which, according to Polya (1954), “must attract the interest of more advanced pupils and free them from monotonous and standardized problems of a textbook”. Also for each of the following problems its place in typology of proofs will be given.

1. The problems in which it is necessary to find a mistake in the proof. Example. Find a mistake in the following transformations. An equality is given

$$\lg\frac{1}{3}=\lg\frac{1}{3};$$

having doubled the left part and leaving the right one without changes, we will get:

$$2\lg\frac{1}{3}>\lg\frac{1}{3};$$

$$\lg\left(\frac{1}{3}\right)^2 > \lg\frac{1}{3};$$

$$\frac{1}{9} > \frac{1}{3}.$$

Where is the mistake? (School level)

In the present case we deal with an example of a problem where the proof is connected only with the use of logic (logical proof).

2. The problems on studying the dependency of the statements A and B: $A \overset{?}{\longleftrightarrow} B$. Example . Find the dependency of the following statements:

monotony of the sequence $\overset{?}{\longleftrightarrow}$ restriction of the sequence,
monotony of the sequence $\overset{?}{\longleftrightarrow}$ convergence of the sequence,
convergence of the sequence $\overset{?}{\longleftrightarrow}$ restriction of the sequence.

(Graduate level)

Proof is connected with the use of intuition (intuitive proof).

3. The problems on constructing examples of various mathematical objects and structures with given conditions. (They have a direct connection with type 2). Example. Construct an example of continuous function without derivative. (The first example of this kind was constructed by Weierstrass). (Graduate level)

Proof is connected with the use of intuition (intuitive proof).

4. The problems on checking various conditions while performing proofs. Example. Check the importance of all conditions in Rolle's proposition.

(Graduate level)

Proof is connected with the use of intuition (intuitive proof).

5. The problems on the search of the dependency between conditions and formation of the *idea of the result*. Examples.

5.1 (simple). Conditions are given: a triangle, a bisector, a circumference circumscribed around the triangle, a circumference circumscribed into the triangle. Study and prove the interdependence between the above conditions.

5.2 (more complex). Conditions are given: a triangle, a point of intersection of the medians of the triangle. What conclusions can be made out of these conditions? In other words, suggest the so-called idea of the result.

5.3 (complex). Conditions are given: a triangle, a trisector of an angle. Suggest the idea of the result. The idea of the result is meant as Morley's proposition here, which presupposes the advanced level of pupils. (School level)

Proof is connected with the use of intuition (intuitive proof).

Classroom situations and typology of proofs for teaching to prove

I would like to give some examples of classroom situations while discussing mathematical reasoning in proving various problems. These classroom situations will be the basis for constructing typology of proofs for teaching to prove.

1.I.M.Glazman (1969) who worked in Kharkov State Pedagogical University named after G.Skovoroda, Ukraine for some time, when listened to the description of a certain structure in infinity-dimensional constructions, usually asked: “And how will it look like in two-dimensional case?” This question often helped to understand the essence of the matter. Thus, it is important to check the possibility of provability in simpler cases and then to go on and evaluate the complexity of the proof in a general case. Let us name such an action while performing proofs, if it is relevant to be used, *a simplified proof*.

2.While reporting the results of his own research, a student introduced a certain class of functions and certain proofs of some propositions for the functions of this class were performed. During the discussion a question arose: is it possible to define at least one function belonging to this class? Nobody taking part in the discussion could give such an example, and the theory itself if not destructed, was seriously damaged. Thus, while performing proofs it is important to check whether the given model exists in general or only under certain restrictions. Let us name such an action while performing proofs, if it is relevant to be used, *an artificial proof*.

3.Sometimes while proving, a situation arises when it seems to pupils (students) that the given conditions are insufficient for its fulfilment. Under such conditions it is often useful to add some complementary statements one by one and to try and perform the proof in such a situation. Thus, it is important to verify to what extent the proof of the problem is performable under these complementary conditions, and then how one can achieve such conditions or whether one can do without them. Let us name such an action while performing proofs, if it is relevant to be used, *a loaded proof*.

4.The following situations while proving certain sentences are of special interest when it is useful to try and prove an inverse proposition, i.e. when the statement of a problem becomes a conclusion and a conclusion, correspondingly, becomes a statement. If an inverse proposition is not true then the given (proposed) counterexample often concentrates in itself those properties that should be paid attention to first of all. Thus, it is important to verify to what extent it is easy to prove the inverse proposition, and what restrictions exist while proving it. Let us name such an action while performing proofs, if it is relevant to be used, *an inverse proof*.

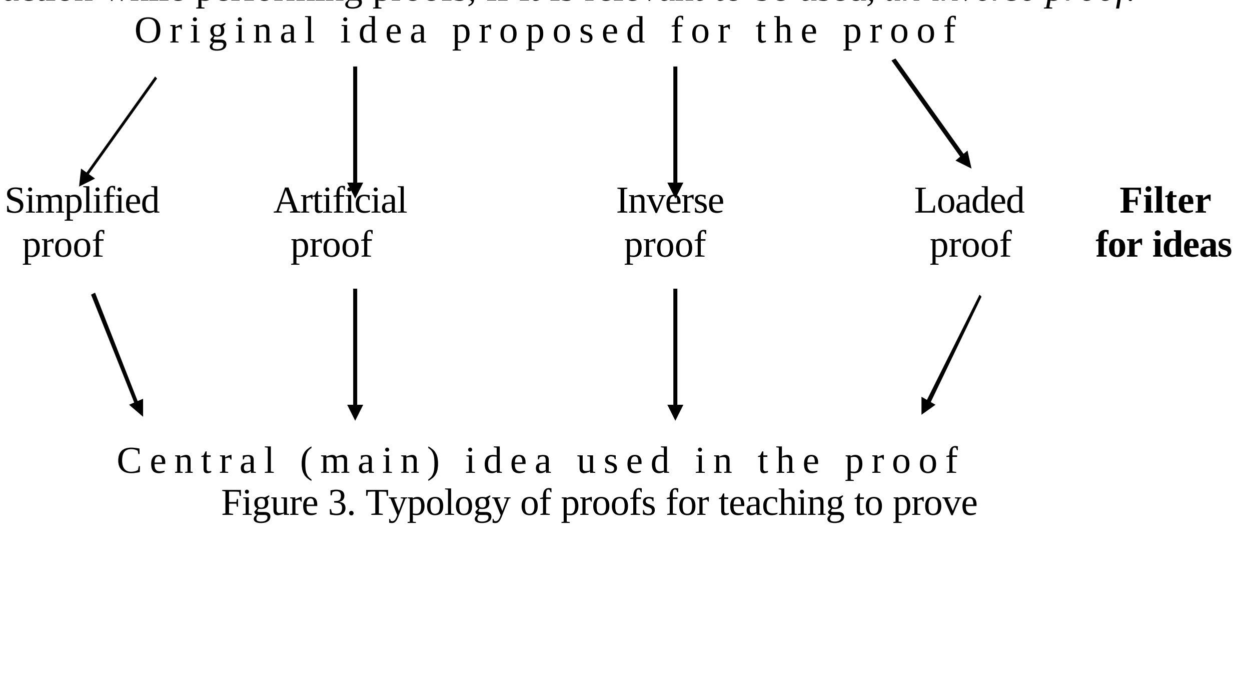

Figure 3. Typology of proofs for teaching to prove

Let us consider the following scheme (Figure 3.). The unity of all actions relevant in this or that case and characterized in classroom situations 1-4, will be called the filter for ideas. An idea for a proof originally proposed by a pupil (a student) passing through the filter, either acquires clear form and becomes a central (main) idea used while proving, or, not passing through the checking of filter, is discarded because of its unsuitability.

Limiting ourselves to some situations and naming the respective actions that may be successfully performed, let us point out that the list of situations may be continued as well as the introduction of corresponding names for the actions used while proving various problems, i.e. the filter is opened for the filling, and the given scheme is only an outline of the typology of proofs for teaching to prove. The most important thing is to teach a student the ability of taking decisions to perform certain actions in one situation and some other actions in another one, the ability of taking decisions to propose one ideas and reject the others. If a student has such an ability (and the above actions are understood as certain mathematical reasoning), it will testify to the fact that this student has elementary foundations of understanding mathematics.

Questions for discussion

Looking at the paper by *Cañadas, Castro, Gómez* (2002) and taking into account the above mentioned, we may suggest to prove one and the same proposition by a teacher (when the proofs exist) using elementary and non-elementary methods (in Graduate level). For example, in 1949 Norse mathematician Selberg gave an elementary proof of the law of prime numbers distribution (without the methods of the theory of complex variable functions), but much more complicated. In this respect a hypothesis arises: a teacher should give the simplest proof understandable to students. The rest depends on students.

Quite recently the term *geometric intuition* has appeared in the works by Atiyah (2001), Fujita and Jones(2002). It is interesting to note that though mentioned above Steklov(1923) provides a geometrical example, he speaks just of intuition.As far as Atiyah's remark on intuition in geometrical form is concerned, we think that in analysis or theory of numbers intuition has always played and is playing not a smaller part, but it is more difficult to present it vividly.

Conclusions

One of the main points in performing proofs by a pupil (student) is his ability to put forward ideas possible to consider. Pupils (students) should be taught on the problems putting forward their own though simplest ideas, later confirming and using them, or refuting and rejecting them. One of the main tasks of mathematical education is that the students should be able not only to perform proofs but to suppose and suggest facts and ideas the true character of which should be cleared out. Generally speaking, an idea and its proof are inseparably connected. In the process of thinking over problems a certain style of thinking is developed; it, together with the store of mastered knowledge, gives a certain level of mathematical education.

Platini, a famous football player, in one of his interviews said: “Leave space for creativity”. He meant that football, like any other sphere invented by the man whether it may be sport, science or culture, is becoming, in its development, various and many-sided. However, these words could be successfully used in students’ mathematical education and become a landmark in improving the process of mathematics teaching both as a whole and in any concrete problem.

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