FORMS AND USES OF ABDUCTION

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This paper offers first steps towards a typology of forms and uses of abductive reasoning in mathematics education. It organises several descriptions of abductive reasoning found in the literature in terms of their logical forms, the relation between specifics and generalities found in them, and the uses suggested for abductive reasoning. Examples are then given of abductive reasoning in mathematics classrooms, and these examples are described making use of the typology developed.

> Most people, if you describe a train of events to them, will tell you what the result would be. They can put those events together in their minds, and argue from them that something will come to pass. There are few people, however, who, if you told them the result, would be able to evolve from their own inner consciousness what the steps were which led up to that result. This power is what I mean when I talk of reasoning backward.
>
> Sherlock Holmes in *Study in Scarlet*

There is a growing literature on the importance of abductive reasoning in mathematics education (e.g., Mason, 1996; Cifarelli, & Sáenz-Ludlow, 1996; Arzarello, Micheletti, Olivero, Robutti, 1998a,b). At the same time there is considerable work being done to clarify the forms and uses of deductive reasoning (which has always been recognised as important) in mathematics education (e. g., Balacheff, 1987; de Villiers, 1990; Hanna, 2000). This suggests that the time may be right to begin to clarify the forms and uses of abductive reasoning. Some steps towards this goal will be taken in this paper.

A typology of the forms and uses of abduction, if it is to be useful in mathematics education, must have a basis in the existing literature on reasoning as well as being applicable to reasoning that is observed in mathematics classrooms. The current usages of the words "abduction" and "abductive" (in logic) are due to Peirce. Prior to his work the word referred to a syllogism in which the minor premise was only probably true, so that the conclusion was only probably true (better known as apagoge). Peirce offers several fundamentally different descriptions of abduction.

In his early work Peirce (c. 1867) emphasises the logical form of abduction. At first he focused on syllogisms and on the role of *characters* of specific cases and classes. A case S ("this ball") might be a member of a class ("the balls in this bag") and have a number of characters ("white" "made of wood" etc.). He described deduction, induction and abduction by the following syllogisms (CP 2.508, 511; 1867):

Deduction	Induction	Abduction
Any M is P	S' S'' S''', etc. taken at random as M's	Any M is, for instance, P' P'' P''', etc.
<u>S is M;</u>	<u>S' S'' S''', etc. are P;</u>	<u>S is P' P'' P'''</u>
S is P	Any M is probably P	S is probably M

S in these syllogisms is the subject, a specific case of interest, and S’ S’’ S’’’ are a number of specific cases. P is the predicate, which describes a character of the subject. P’ P’’ P’’’ are a number of characters. Thus, the difference between induction and abduction in this formulation has to do with whether a number of cases are found to share a character, leading to the conclusion that all similar cases share that character (by induction); or if a single case is found to share a number of characters with a class of cases, leading to the conclusion that the case belongs to that class (by abduction).

Using these syllogisms Peirce tried to distinguish between two kinds of reasoning from cases: induction and abduction. As induction is often defined simply as discovering general laws by examining specific cases, it is sometimes difficult to make the distinction between it and abduction. We will see that in Peirce’s later descriptions of abduction, he explored others ways to maker this distinction.

Later (c. 1978) Peirce rephrased his descriptions in terms of rules, cases and results, and this description is often quoted (e.g., Arzarello et al., 1998ab; Mason, 1996). His description is summarised in these syllogisms (based on CP 2.623; 1878)

Deduction

Rule – All the beans from this bag are white	M is P
<u>*Case* – These beans are from this bag</u>	<u>S is M</u>
Result – These beans are white	S is P

Induction

Case – These beans are from this bag	S is M
<u>*Result* – These beans are white</u>	<u>S is P</u>
Rule – All the beans from this bag are white	M is P

Abduction (Hypothesis)

Rule – All the beans from this bag are white	M is P
<u>*Result* – These beans are white</u>	<u>S is P</u>
Case – These beans are from this bag	S is M

The differences between this formulation and Peirce’s formulation of 1867 are slight, but significant. Instead of “These beans” sharing a number of characters, only one character “being white” is involved in this canonical example. This suggests that Peirce saw abduction as possible on very limited evidence, perhaps because in examining instances of abduction in scientific discovery, he encountered such situations. As well the specific cases S’ S’’ S’’’ enumerated in his 1867 formulation are now subsumed under a single subject “these beans”. The specific nature of the cases is thus downplayed, allowing for abduction in which the “case” is in fact a generality (in fact “These beans are white” could be phrased as “All the beans in this sample are white”, which has the same generality as “All the beans from this bag are white”).

Before considering Peirce's final formulation of abduction, I will refer to Eco's work on abduction, which is based on the formulation above.

Eco (1983) describes abduction as the search for a general rule from which a specific case would follow. He identifies (as do Bonfantini and Proni, 1983, using different terms) three kinds of abduction. Given a specific case, the reasoner may be aware of only one general rule from which that case would follow. This Eco calls "Hypothesis or overcoded abduction" (p. 206). If there are multiple general rules to be selected from, Eco calls the abduction "undercoded abduction" (p. 206). It can happen that there is no general rule known to the reasoner that would imply the specific case in question. Then the reasoner must invent a new general rule. This act of invention can also occur when there are general rules known that would lead to the specific case, but they might be unsatisfactory for some reason. An abduction that involves the invention of a new general rule Eco calls "creative abduction".

The logical forms identified by Peirce and Eco do not exhaust the possibilities. Artful manipulation of the premises can reduce many abductions to these forms, but it is not clear that forcing descriptions of reasoning into predetermined forms is useful for research in mathematics education. In addition, there are some cases where such manipulations may not be possible at all. For example, as Mason (1996, p.37) points out, "the tricky part about abduction is locating at the same time the appropriate rule and the conjectured case." In other words, it might occur that the *only* premise of an abduction is the result, and that *both* the rule and the case are consequences of it.

While Peirce, in his early work, emphasised the differences in logical form between deductive, inductive, and abductive reasoning, in his later work (e.g., "Lectures on Pragmatism", CP 5.14-212, 1903) he shifts the emphasis to the function satisfied by each kind of reasoning. The logical form of abduction has been reduced to:

C
<u>A implies C</u>
A

Abduction for Peirce has become identified with "explanatory hypothesis" and his criteria for an abduction to be a good abduction has come to include, at least, that it "must explain the facts" (CP 5.197; 1903). It has become part of a process of inquiry in which abduction, induction, and deduction play particular roles. Abduction explains by introducing a new rule, deduction draws necessary conclusions from the consequent of the abduction, induction evaluates the consequent by comparing the conclusions draw from it to experience (CP 6.469-476; 1908). I rephrase this, in the terms I am accustomed to using for the needs satisfied by mathematical reasoning (see, e.g., Reid, 1995, 1996), as: *abduction explains and explores; induction and deduction verify.* The exact uses Peirce ascribes to abduction, induction and deduction are not as important as the shift in his writing from emphasis on logical form to emphasis on describing reasoning by the uses (functions, purposes, needs) addressed by reasoning. This can lead to confusion, as it is far from clear that each type of reasoning is always associated with the same uses, or that each use is addressed by only one type of reasoning. In fact, given the recent work in mathematics education demonstrating that deductive reasoning has many different uses (e.g., de Villiers, 1990; Hanna, 2000) Peirce's associations between deduction, induction and verification, and between abduction and explanation may be more limiting than they are useful.

In all Peirce described abduction in three distinct ways, drawing on different aspects of reasoning in his descriptions and identifying different issues as central. In his descriptions three issues are evident: Logical form, specific versus general cases, and the use of reasoning. In the following examples these issues will be considered in specific descriptions of mathematical activity involving reasoning that I would call abductive.

Two cases of abduction

In this example a teacher in France is guiding a level 4 (age 13-14) class through a proof of the Pythagorean Theorem. They have concluded that ABCD is a rhombus, because it has four equal sides (see figure 1). The transcript is from Knipping (2002). The translation is mine. The original is in the appendix.

1 T: It's a rhombus. That's a sufficient condition to have a rhombus. There is no need for anything else. … But that's not what I asked you to prove. I asked you to prove that it's a …

2 S: Square

3 T: Square. So, under what condition is a square, uh, is a rhombus a square?

4 Ss: If it has a right angle

5 T: If it has …?

6 Ss: A right angle

7 T: A right angle, that's enough.

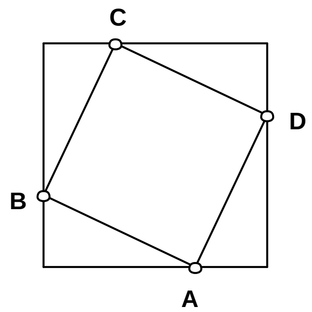


Figure 1

Here the abduction is:

Case – ABCD is a rhombus
Result – ABCD is a square
<u>*Rule* – If a rhombus has a right angle then it is a square</u>
Case – ABCD has a right angle

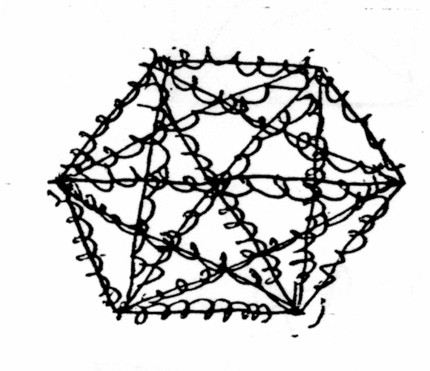
There are several things of interest about the logical form of this abduction. One is the presence of the first premise. Because of Peirce's habit of structuring abductions as syllogisms with two premises, this form would not have been used by him. It is also interesting to reflect on the truth value of the various premises. The Case and the Rule have been established, and the Result is suggested by the diagram, as well as by the teacher's injunction to prove it, as a normal part of the didactic contract is that statements the teacher asks students to prove should be true. Otherwise this abduction is more or less similar in form to the form Peirce described in 1878. In

Eco's terms this is an overcoded abduction as there are many rules the students could have invoked. For example, *If the diagonals of a rhombus are equal, then it is a square,* or *If two adjacent angles of a rhombus are congruent then it is a square.* Either of these rules would have led to conclusions which might or might not have been helpful to the students in proving that ABCD is a square.

Concerning the specificity or generality of the premises and conclusion in this abduction, all but the Rule seem to be specific to ABCD. In this the abduction is closer in form to Peirce's formulation of 1867. A further complication is the role this abduction will later play as part of a general proof of the Pythagorean theorem. The quadrilateral ABCD is in fact a "generic example" (Balacheff, 1987) being used in a general proof. Therefore all the specific features of it are actually general.

Turning finally to the use of this abduction, it is not, as Peirce's later comments on abduction would have it, intended to explain why ABCD is a square. Rather its use is in exploring what features of ABCD need to be established in order to create a deductive argument for the truth of the theorem that will satisfy the teacher's needs. Recall that for Peirce the use of abduction is explaining surprising cases. The fact that ABCD is a square is hardly surprising. That this example fits Peirce's early descriptions of abduction but not his later description could lead us to conclude that this is not really an example of abduction at all. Instead I would suggest that what it tells us is that Peirce's different descriptions are all useful in identifying issues central to describing reasoning, but they are not always compatible with each other, nor complete by themselves.

My second example of abduction occurred when Jason, Nicola and Sofia, three Canadian grade 8 students (age 13-14) were solving the problem of determining the number of handshakes that occur when *n* people shake hands. They were first asked to explain why 15 handshakes occur when 6 people shake hands, which they did using a diagram (see figure 2). They were then asked to determine the number of handshakes for 26 and 300 people. They solved the case of 26 people by adding 25+24+…+2+1. As Jason added the numbers using a calculator, Sofia claimed to "know an easier way". She tried to explain her way using a diagram (see figure 3) but Jason protested "How would you write that down so you wouldn't have to draw all those lines? Is there a way?"



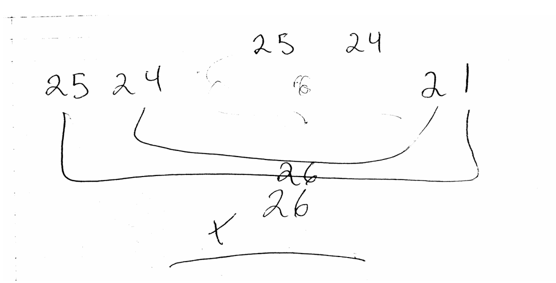


Figure 2: diagram for n=6 Figure 3: Sofia's diagram

Sofia tried to show how to use her method to solve the case of 300 people, proposing 150 times 300 as the answer. Jason suggested seeing if it worked for 26. Sofia said it should be 26 times 13, which Jason calculated, getting 338, not 325 as he had obtained by adding. He checked his addition, once more obtaining 325, "So that can't be right. But you were close." Sofia began guessing at other numbers near 13 and 26. This transcript begins after they had tried 14 and 13.5.

Time: 1:04:51

Sofia: Well, it was just a guess, ok?

Jason: Well you see that would be a problem.

Nicola: Yeah but you can't use that because13 and 13 is 26, and 13.5 and 13.5 is 27.

Jason: OK, let's see…

Sofia: Umm, Oooh, Oooh! [Time: 1:05:11]

Jason: Umm, I know, maybe it's… just a second, just a second..

Sofia: Wait, wait, nono, it's 27, it's 27! [Time: 1:05:22]

Jason: Maybe it's the number times half the number, umm, subtract half the number

Sofia: You lost me

Jason: Because that would work, 325 subtract 13, which is half of 26 is right.

Sofia: Try it again.

Jason has used abductive reasoning to arrive at the general rule

> [The number of handshakes is] the number [of people] times half the number, subtract half the number

from the specific case:

> Because that would work, [the number of handshakes for 26 people is] 325 [which is 338] subtract 13, which is half of 26, is right

This can be formulated as Peirce's formulated abduction in 1867, if it is expressed as:

> Any formula for H(n) gives the answer for the case n, and is based on the value of n
>
> <u>26×(26/2) – (26/2) gives the answer for the case 26, and is based on the value of 26</u>
>
> $n \times \frac{n}{2} - \frac{n}{2}$ is probably a formula for H(n)

As this formulation would suggest, the premises list a number (two here) of characters of a formula, and then assert that the specific subject in question has those characters, leading to the conclusion that the subject is probably a formula. In Eco's terms it is a creative abduction, as Jason has invented a general rule to account for the unexpected near success of Sofia's method.

This abduction is used (as the later Peirce would suggest) to explore (in finding a formula) and to explain (why Sofia's method gave an answer that was "close"). The next step in Peirce's sequence of uses is a deductive specialisation of the conclusion of the abduction to another specific case, which is in fact what happened.

After proposing testing his conjecture for 300 people, then 100 people, Jason settled on 10 as a good number.

[Time: 1:06:20]

Jason: Ok let's try it with 13, or, no, no, 10, let's try it with 10. 10 times 5 equals 50,

Sofia: Try it with 20, or whatever number…

Jason: ok, then you would subtract five because that's half of ten right? [time: 1:06:40] Alright.

Nicola: Yeah, that's 45.

Jason: So that's what we got 45.

The next step is an inductive test of this result. Jason made errors in his adding however, first skipping 10, then adding it twice.

Jason: 35?

Sofia: [Laughing hysterically]

Jason: Dammit! I forgot to push 10! Shut up! 10+9+8+…+1 equals 55? [Time: 1:07:55] OK, so if the number was 10…

Sofia: Can I try something different?

Jason: I'm positive this has to be right though. The number was ten.

This abduction then can be described as a creative abduction having the form described by Peirce in 1867, and being used according to the pattern he set forth in the early 1900s. This does not make it a better example of what Peirce meant by abduction than the previous example, however. It only points out that his early and late descriptions *can* be useful in combination, though they are not always. In addition I would question whether declaring one example to fit Peirce's descriptions and the other not to fit it would be a useful conclusion for mathematics education research. Our goal, after all, is to describe the reasoning of students in mathematics classrooms, not to make it fit predetermined categories from philosophy. Peirce's work is useful not because it tells us what categories to fit students' reasoning into, but because examining Peirce's descriptions allows us to identify issues of importance in describing reasoning.

Conclusion

Abductive reasoning is an important part of mathematical reasoning, and the development of students' abductive reasoning should be a concern for mathematics educators. As a first step towards supporting that development we must come to a better understanding of the forms of abductive reasoning students use, and the uses they have for abductive reasoning. No single description of abductive reasoning found in the philosophical literature is sufficient to allow this; elements of several are needed. And other sources beyond those considered here might inform the development of a typology of abduction; for example, the heuristic syllogisms described by Polya (1968/1990).

The typology I have outlined in this paper is a step towards being able to describe students' reasoning in its own terms, rather than in terms of predetermined categories developed for other purposes. There must necessarily be a co-evolution of such typologies and studies of students' reasoning, as our language affects what we can see, while what we see affects our language. Perhaps the most important element of this research will be coming to a better understanding of the ability to reason abductively, what Peirce calls the "guessing instinct", and the degree to which such an "instinct" might be natural (as Peirce himself suggested) or derived from our cultures (as Bonfantini and Proni 1983: 134, suggest).

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Appendix

1 P : «C’est un losange.» C’est une condition suffisante pour avoir un losange, on n’a pas besoin d’avois autre chose … Alors ce n’est pas ce que je vous avais demandé de démontrer, je vous avais demandé de démontrer que c’était un …

2 E : Carré

3 P : Carré. Alors, à quelle condition un carré, eh un losange est-il un carré?

4 Es : S’il a, s’il a un angle droit

5 P : S’il a … ?

6 Es : Un angle droit.

7 P : Un angle droit, ça suffit.