On the Role of "Grundvorstellungen" for Reality-Related Proofs – Examples and Reflections

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There are a lot of arguments for the inclusion of applications and modelling in mathematics teaching: so-called pragmatic, formal, cultural, and psychological arguments. Among the psychological arguments, the most often mentioned is the role of applications for motivating and introducing new topics and for practising and consolidating them. What is rarely mentioned is another psychological aspect: applications provide contexts for what I call reality-related proofs. This is the topic of my paper. It has three aims:

1) *to explain the concept of reality-related proof by means of four examples,*
2) *to elaborate the role of Grundvorstellungen in these proofs,*
3) *to show why all this can be important for mathematics teaching.*

I concentrate deliberately on theoretical considerations and do not refer to empirical aspects.

1. An introductory example

Example 1: Let us presuppose a pupil knows the definition of

$$\binom{n}{k} := \frac{n(n-k)\cdots(n-k+1)}{k!} \quad (0 \le k \le n)$$

as well as its interpretation as a number in certain real contexts. One instance might be where, in a group of 11 friends, 4 of them are to be chosen by lot for a committee to prepare the Christmas party. How many different committees are possible? That is, as we know, $\binom{11}{4}$, since there are $11 \cdot 10 \cdot 9 \cdot 8$ different arrangements of 4 persons, and 4! of these, respectively, lead to the same committee.

Let us further assume that, by calculating some numerical examples, the pupil finds that in all these cases $\binom{n}{k} = \binom{n}{n-k}$. Is this always true?

THEOREM 1: $\boxed{\binom{n}{k} = \binom{n}{n-k} \quad \text{for all } k, n \ (0 \le k \le n)}$

How can it be proved?

The assertion means in the same real context as just stated: With 11 persons, there are as many committees consisting of 4 persons as there are committees of 7. The basic idea for proving this is, as we know, very simple. Every committee of 4 corresponds to a (non-)committee of 7 and vice versa, for in each case we just take the remaining persons (Fig. 1).

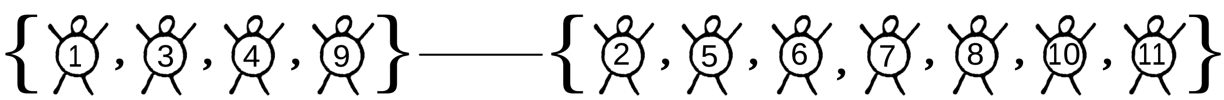

Fig. 1

That's it!

If the pupil should not see this correspondence immediately, we could argue in full detail as follows (but this is not so relevant).
We realise mentally all arrangements of 4 persons and put these together in groups of 4! arrangements each. Thus we get the $\binom{11}{4}$ possible committees of 4. We do the same with all arrangements of 7 persons. Now we look at every single arrangement to see which persons are *left,* and we add these (mentally), in all possible arrangements (Fig. 2).

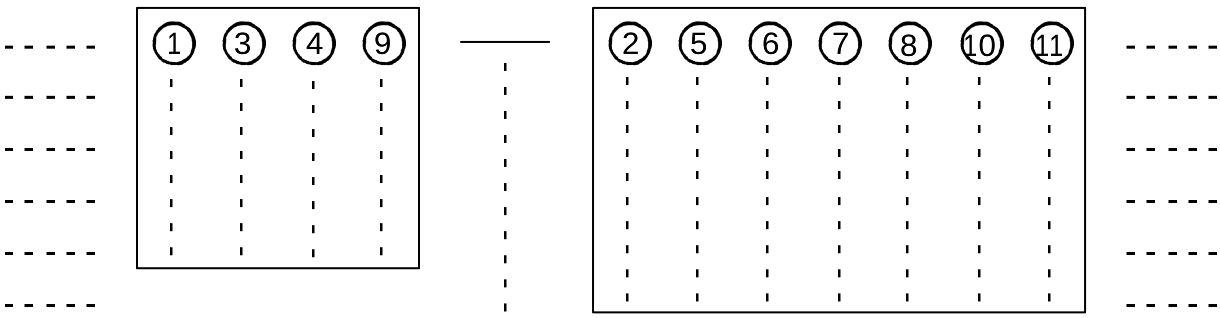

Fig. 2

Thus, by combining everything, we obviously get all 11! permutations of 11 persons.
Now, in the first place, it is clear (using the same bijection as before) that there are as many committees of 4 as committees of 7 and, in the second place, these numbers can obviously be calculated as follows:

$$\binom{11}{4} = \text{number of committees of } 4 = \frac{\frac{11!}{7!}}{4!}$$

$$\binom{11}{7} = \text{number of committees of } 7 = \frac{\frac{11!}{4!}}{7!}.$$

So the equality can also be seen formally.
To this detailed contextual argumentation (which, as I said, is not necessary if the pupil is sufficiently familiar with the real context), the following well-known formal-mathematical argument corresponds:

$$\binom{n}{k} = \frac{n(n-k)\cdots(n-k+1)}{k!} = \frac{n!}{k!(n-k)!}$$

$$\binom{n}{n-k} = \frac{n(n-1)\cdots(k+1)}{(n-k)!} = \frac{n!}{k!(n-k)!}, \text{ so}$$

$$\binom{n}{k} = \binom{n}{n-k}.$$

Admittedly, this is formally trivial but, taken as such, it gives insight only on a higher mathematical level (symmetry!).

2. On the concept of reality-related proving and its significance for teaching

What have we just done in example 1?

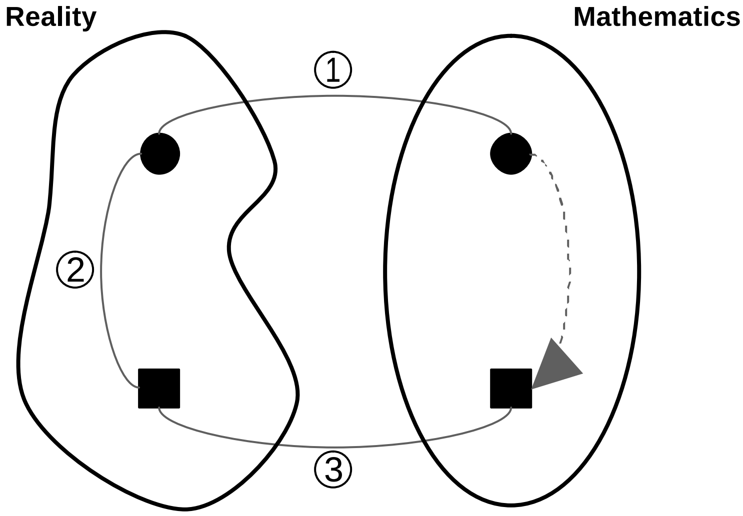

Fig. 3

Here (Fig. 3), in a very simple model, we have Mathematics and the Rest of the world, in short: Reality (note that this is a rather broad notion of reality, including artificially dressed-up or constructed contexts as well). First ? we have *interpreted* the premises (certain mathematical objects or operations and certain interrelations) in a specific real context, we have – as I call it – *realised* them. Second ? we have carried out certain *arguments* or *actions* within this context by means of contextualized knowledge. This has led to certain results. Third ? we have *translated* these results *back* into mathematics and hereby obtained mathematical results. Altogether we have thus proven a certain mathematical theorem.

That's what I call a *reality-related proof* of this theorem. Sometimes I also use the term *contextual proof.*

Note that this cycle "realisation-argumentation-mathematisation" is just the *reverse* of the usual modelling cycle.

So a reality-related proof is – in short – a chain of certain correct conclusions based on certain valid premises, where conclusions and premises are realised in a specific context. Some of the conclusions may consist of certain actions, actually carried out or only imagined, accompanied by reflections upon the validity of the actions. All conclusion must be capable of being *generalised* directly from the concrete case, so that case has to be "generic". If formalised, they have to correspond to correct formal-mathematical arguments. It is, however, not necessary for such a formalisation to be actually effected or even recognisable.

The kind and extent of the conclusions depend heavily on the specific preknowledge in the given real context. This may vary individually (see example 1). Therefore there are different levels in step ? . Sometimes the real context may be so familiar that after step ? , realisation, immediate insight is possible, which means step ? does not contain any arguments. It is subject to discussion whether one should speak of a "proof" in this case. **Example 2** (just as well-known as example 1) is an example of that kind:

THEOREM 2: $\boxed{f' = 0 \text{ in } I \Rightarrow f = \text{const in } I}$

Proof: Step ? : we interpret

x – time

$f(x)$ – distance covered by a vehicle

We presuppose as known that then

$f'(x)$ – instantaneous velocity (what the speedometer shows)

Step ? : from everyday knowledge it is clear that if the speedometer has been showing zero all the time then the car has always been standing still.

Step ? : translation back into mathematics yields the theorem.

A reality-related proof - with reference to a certain basis of argumentation - is completely *valid*! It is only codified in a non-formal way, it is - as Blum and Kirsch (1991) have called it - *pre-formal*. There are other kinds of pre-formal proofs, for instance what we call geometric-intuitive proofs (for examples see also Kirsch, 1979, and Wittmann and Müller, 1990).

By the way, from a philosophical perspective it is actually the other way round: By regarding certain premises as true and certain conclusions as admissible we *define* our concepts of rigorous (pre-formal) proof and of truth. The concept of pre-formal proving may constitute a sound philosophical basis for school mathematics.

Why are pre-formal proofs so important for learning and teaching? For several reasons, "to know proofs" and, more than that, “to be able to prove” belong to the important goals of mathematics instruction; proofs and proving are a characteristic feature of mathematics (see Hanna and Jahnke, 1996, ch. 3 and 5, for a survey of the didactical role of proofs). *Formal* proofs are mostly the final stage in a genetic development - historically as well as epistemologically as well as psychologically. In preceding stages, from grade 1 on, valid proofs are accessible to learners if these proofs are just represented appropriately, corresponding to the stage of development of the individual cognitive structures. This is provided by suitable *pre-formal* proofs. Such proofs can also be much better kept in mind by the pupils.

I ought to mention that there are also fundamental *problems* with the use of pre-formal proofs in mathematics teaching, for instance: Who is to judge whether a certain pre-formal argument is correct, whether a certain proof is valid? How can a pupil realise if a conclusion is incorrect? For an example of this problem and for more reflections on it see Blum and Kirsch (1991). In this paper I concentrate on “positive” examples.

It is necessary to remark that the idea of non-formal proving is not new, of course. What *is* presumably *new* is our view of this concept and of its role in teaching, especially of reality-related proving, and our way of presenting some of the examples.

3. On the role of “Grundvorstellungen”

Back to reality-related proofs in particular. A decisive question is the following: what cognitive structures must be available so that such proofs can be carried out? To put it another way: how have the translations between mathematics and reality in steps ① and ③, as well as the conclusions in step ②, in our examples been possible? For this, what is absolutely necessary are appropriate reality-related “concept images”, “intuitions”, “fundamental notions”, (in German:) *Grundvorstellungen* (abbreviation: GV) of the involved mathematical objects, operations and relations, *realisable* in the real context in question.

In example 1 we used:

– GV “product of natural numbers as a number of possibilities“, more concretely possibilities of arrangements of certain real objects;
– GV “dividing as partitioning“ of certain objects;
– GV “variable as a placeholder“ for certain objects;

and, on a higher level,

– GV “binomial coefficient as a number“ of combinations of certain objects.

In example 2 we used:

– GV “variable as a varying quantity“;

– GV “real function as a 1-1 mapping“ between certain quantities, concretely as a distance-time relation;

and, on a higher level,

– GV “derivative as a rate of change“ of certain quantities, concretely as instantaneous velocity.

What is, actually, a *GV* of a mathematical topic? I use this concept in the way we have developed it in Kassel during the last few years (see especially the Ph D thesis Hofe, 1995). There is no space here to elaborate on this. Very roughly speaking, GVs describe relations between mathematical topics, real contexts and individual mental structures. They carry the *meaning* (in German *Bedeutung)* of a mathematical topic and, to the learner, they represent the “essential“, the “heart“ of the topic. To be a bit more precise: they serve

– to constitute *meaning* (in German *Sinn),*
– to construct *mental representations* which also allow for actions in the imagination,
– to create *links* to the real world and thus to enable individuals to translate between mathematics and reality.

We distinguish between two different aspects:

– *normative* (prescriptive): GVs describe what learners *ought to* acquire - what we also call “basic ideas“ of topics.
– *descriptive:* GVs describe what learners *have actually* acquired - what we also call “individual images“ of topics (sometimes these may include *Fehlvorstellungen,* misconceptions, wrong intuitions as well).

Generally, there are *several* GVs of a given mathematical topic. Here are three examples.

Product of two natural numbers:

– “repeated addition“-GV
– “number of pairs“-GV

Fractional number:

– “part-whole“-GV (3/4 as a portion: 3 out of 4 parts)
– “operator“-GV (a given quantity is transformed into "3/4 of" this quantity)
– “ratio“-GV (3/4 as a relation between 3 parts and 4 parts)

Function:

– “mapping“-GV
– “covariation“-GV
– “object“-GV

GVs are, as I said, carriers of meaning. If we regard *understanding* as the process of grasping the meaning - as Sierpinska (1994) proposes - then GVs are crucial, are necessary for real understanding. Establishing a network of appropriate GVs with pupils is, in my view, the most important task of mathematics teaching from grade 1 on.
In particular, if learners are really to understand mathematical facts and their proofs then they definitely have to acquire appropriate GVs. If this is the case one gets proofs that explain and not only proofs that prove, one gets proofs that give answers to the question of *"why* is it true?" and not only to *"is* it true?" (this distinction has been emphasised by Hanna, 1990, among others), one gets *semantic* proofs in the sense of Knipping (2002). If understanding is an essential aim of mathematics teaching, and if one regards explaining as the most important purpose of proving as does Hersh (1993), a view I share, then pre-formal and particularly reality-related proofs gain a crucial significance for mathematics teaching and learning. They are not merely – pedagogically – a clever device for making theorems and their proofs accessible early (as has been emphasized in section 2), not merely a temporary stage on the way to formal proofs, but rather – epistemologically – an appropriate means for revealing the meaning of certain mathematical facts.

In order to avoid misinterpretations I would like to emphasise first that I do not claim pre-formal proving to be the only way to understanding and second that formal proofs remain absolutely relevant for learning and teaching, too. In particular, formalising preformal proofs and studying connections between pre-formal and formal proofs certainly contribute to understanding on a higher (and, at least for pupils at schools, very demanding) level.
Now two more examples to illustrate the concept of reality-related proving and the role of GVs therein.

4. The "Schorle" proof

Example 3: What is well-known is the wrong strategy of pupils when adding fractions: "numerator plus numerator and denominator plus denominator", e.g. "$\frac{3}{4}+\frac{7}{9}=\frac{10}{13}$". According to the educational principle of handling pupils' mistakes in a positive and constructive way, this unusual kind of addition can be the starting point for reflections: what do we really get by doing this? This has been treated with grade 7 pupils by my colleague Johannes Schornstein (for recent instruction experiences with this example see the case study described by Biermann and Blum, 2002). The pupils found, by calculating some examples, that this peculiar "sum" seems to lie always between the two fractions; written out formally:

THEOREM 3: $\boxed{\frac{a}{b}<\frac{c}{d}\Rightarrow\frac{a}{b}<\frac{a+c}{b+d}<\frac{c}{d}\quad\left(a,b,c,d\in\mathbb{N}^{+}\right)}$

It was an exciting question for the pupils: is this really always true, and how can we understand it?

Proof: Let's take $\frac{3}{4}$ and $\frac{7}{9}$ as an example.

We activate the ratio-GV of fractional numbers and interpret $\frac{3}{4}$ as a mixture of 3 parts wine and 4 parts mineral water, and the same with $\frac{7}{9}$. Such a mixture of wine and water is in South Germany (where Johannes and I come from) called a Schorle. We assume all parts to be of the same size.
What shall we use in the following? We know from everyday experience with (idealised) Schorles: The mixture ratio defines the "wininess" of a Schorle, which can be tasted (or seen, especially if it is red wine). Two Schorles have the same wininess (that is the same taste or colour) if and only if the fractions are equivalent; that means proportional enlargement of the two components of a Schorle doesn't change the wininess. Schorle 1 is less "winy" that Schorle 2 if and only if fraction 1 is less than fraction 2.

If we want to compare the two given fractional numbers we find $\frac{3}{4}=\frac{6\frac{3}{4}}{9}$ and hence $\frac{3}{4}<\frac{7}{9}$ since $6\frac{3}{4}<7$. So the $\frac{3}{4}$-Schorle is less winy than the $\frac{7}{9}$-Schorle.
Now we pour the two Schorles together (Fig. 4; of course, real Schorles look a bit different!).

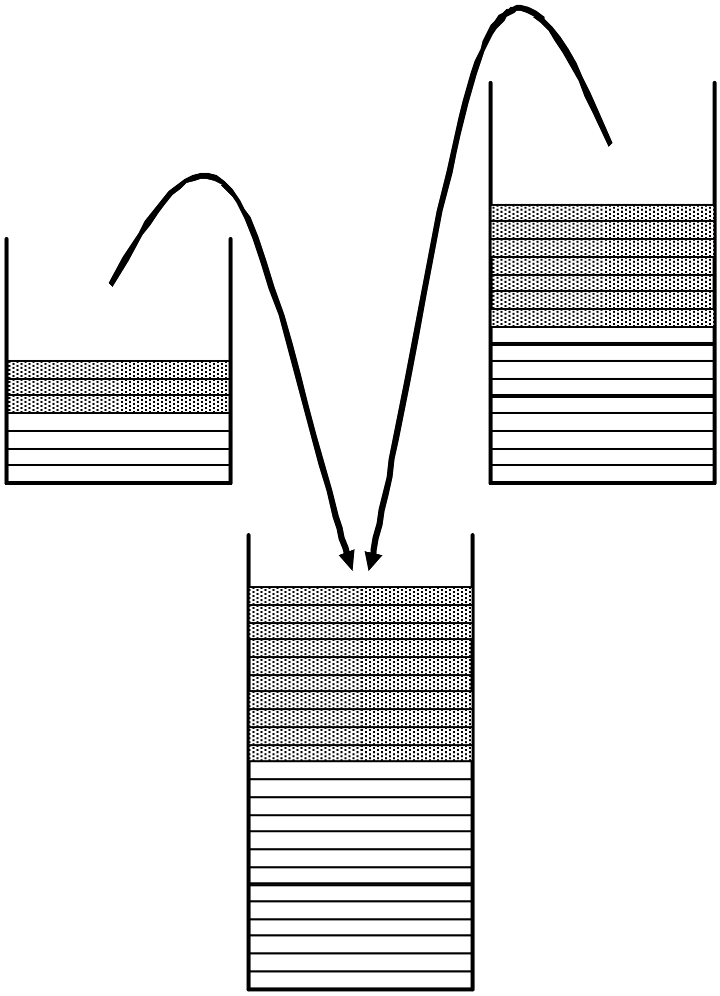

Fig. 4

We get a Schorle with $3+7=10$ parts wine and $4+9=13$ parts water. Here this peculiar "addition" makes sense!
Now it should be clear from everyday experience with mixture ratios (with respect to its wininess) that the mixed Schorle truly lies *between* the two initial Schorles; that is it tastes a bit more winy than the one and a bit less winy than the other. Re-translation and generalisation result in our assertion!
How can we explain this experience? We could argue in a more detailed way as follows.
It is clear that if we pour together two Schorles with the same wininess then the mixed Schorle has also the same wininess, for instance $\frac{3}{4}$ and $\frac{6.75}{9}$.
If we take the $\frac{7}{9}$-Schorle instead of the $\frac{6.75}{9}$-Schorle and mix it with the $\frac{3}{4}$-Schorle then we obviously add a bit more wine, so the new mixed Schorle tastes a bit more winy than the first one. That's it!

What GVs have we used in this proof?
– ratio-GV of fractions (concretely as Schorle mixtures),
– GV of equivalent fractions (as Schorle mixture ratios),
– GV of "<" for fractions (as mixture ratios),
– GV of "<" and of "+" for positive rational numbers (concretely as volumes of liquids).

We can translate the detailed arguing directly into formal mathematics.

Let $\frac{a}{b}<\frac{c}{d}$. Determine c' so that $\frac{c'}{d}=\frac{a}{b}$.

Then obviously $c'<c$ and $c'=ra,\ d=rb$ for a certain $r>0$.

Then $\frac{a}{b}=\frac{(1+r)a}{(1+r)b}=\frac{a+c'}{b+d}<\frac{a+c}{b+d}$.

That's it! (Second part analogously)

Note that we have obtained a *new* proof for this inequality, for usually it is proven purely formally by "multiplying the denominators up" and so on. The usual formal proof is more familiar to mathematicians, but it yields no insight at all, whereas the reality-related proof gives real insight. This example again supports our thesis from section 3: real understanding is only possible when working with GVs.

Note that, in example 3 and likewise in examples 1 and 2, we have *subsequently* proven a previously given assertion. However, in all examples it would have been equally possible to argue within the context without a given assertion and to find it, to *discover* it, or perhaps we should better say to *create* it – and, best of all, to have this done by pupils on their own.

A final comment on example 3: a reality-related proof such as the Schorle proof consists of three steps (see section 2), realising, contextual arguing, mathematizing. Here and in all examples, the mathematical result has to be independent of the specific context. Taking into consideration the well-known *context dependence* when learning and using mathematics, this is only possible if the relevant GVs have already been built up in the individual *before*, and need only to be activated in a specific context. In all examples we have assumed this – these are high demands indeed of mathematics teaching!

We also assume this in the following and final example. Here we are going to deal with the GVs of derivative and integral. These are supposed to be generated already:

– derivative as a rate of change,
– integral as a "generalised product", that is a limit of sums of products of quantities.

Now, in this example, a theorem is *discovered* by contextual reasoning (see Blum and Kirsch, 1996). I myself have taught this example in a grade 12 class (basic course), that means the students discovered the theorem by themselves, guided by the teacher, of course.

5. Discovering an important theorem

Example 4: Given the derivative $f(x)=\frac{dG(x)}{dx}$, what does the integral $\int_a^b f(x)dx$ then mean?

We interpret again

x – time

$G(x)$ – distance covered by a vehicle

and hence

$f(x)$ – instantaneous velocity of the vehicle.

The pupils know that, for small pieces of time, we have

$$\frac{\Delta\text{distance}}{\Delta\text{time}}\approx \text{instantaneous velocity (at that time)}$$

or $\Delta\text{distance}\approx\text{velocity}\cdot\Delta\text{time}$

This (idea of linear approximation) holds for sufficiently small Δtime as accurately as desired.

We now calculate the "generalised product" of velocity and time. The pupils know that this is done in four steps. First we divide the given time interval into small pieces. Second we regard the velocity as constant in each sub-interval and calculate the products, there; we get

$$\text{velocity} \cdot \Delta\text{time} \approx \Delta\text{distance}$$

Third we sum up all these products: $\sum \text{velocity} \cdot \Delta\text{time}$.

It is clear that this sum approximately equals the total distance travelled in the time interval. Fourth we let the number of sub-intervals increase beyond any limit and their lengths $\Delta\text{time} \to 0$. Now it's absolutely clear that the result of this process, that is the generalised product of velocity and time, is equal to the global distance travelled (difference in displacement).

By the way, this was, on the whole, the way that Evangelista Torricelli argued as early as the first half of the 17th century!

Re-translation and formalisation results in:

THEOREM 4: $\boxed{\int_a^b \frac{dG(x)}{dx} dx = G(b) - G(a)}$

This is nothing else than the second fundamental theorem of calculus! (A supplementary analysis of the proof shows that *f* has to be continuous.)

Usually in calculus teaching, this theorem is formally deduced as a corollary to the first fundamental theorem of calculus. Thus its meaning is reduced to a mere formula for calculating integrals. A reality-related proof such as ours enables pupils genuinely to understand the theorem and reveals so-to-speak its "true" meaning: the integral of a rate of change function (the "total effect" of the rates of change) on an interval is the increase of the original function there (that is "integrating as reconstructing"). By the way, a purely geometrical argumentation (derivative as slope, integral as area) will not be able to reveal this meaning (see Blum and Kirsch, 1996, for a more detailed analysis).

In example 4 we found a theorem by certain contextual conclusions. As I said before we could have done so in all examples (for examples at the primary school level see Wittmann, 1996). This procedure brings to mind the so-called *operative principle* of learning mathematics: "what happens with … if …". Therefore I dare now to formulate the

Operative principle of reality-related proving:

Take any mathematical topics, translate them into some real context, then – on the basis of contextual knowledge – carry out any correct arguments (or actions) whatsoever, and lastly translate the results back into mathematics. Then you have obtained a certain – more or less interesting – mathematical theorem.

6. Conclusion

In mathematics instruction orientated towards understanding pupils are to develop appropriate GVs from the beginning – an ambitious and never-ending task. GVs enable pupils to translate between reality and mathematics, also and especially they contribute to proving. By means of GVs, proofs become accessible that enable pupils to gain non-formal insight and help to really understand.

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