

LOGIC AND MATHEMATICAL REASONING FROM A DIDACTICAL POINT OF VIEW A MODEL-THEORETIC APPROACH

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Abstract : In this paper, we intend to show, on various examples, the relevance of predicate calculus, and specially the model-theoretic approach introduced by Tarski, for a didactical analyse of mathematical reasoning and proofs. The main interest of this framework is to help us in elucidating the relationship between syntax, semantic and pragmatic as defined by Morris, allowing consequently to consider rigorously how the knowledge of pupils and students may modify their reasoning.

Résumé : Dans cette communication, nous nous proposons de montrer, à partir d'exemples variés, la pertinence du calcul des prédicats et principalement de l'approche modèle-théorique pour une analyse didactique du raisonnement et des preuves en mathématiques. L'intérêt principal de ce cadre théorique est de nous aider à élucider les relations mutuelles entre la syntaxe, la sémantique et la pragmatique, entendues au sens de Morris, permettant en particulier de prendre en compte de manière rigoureuse l'état des connaissances des élèves et des étudiants.

Introduction

As it is well known, most of pupils and students meet strong difficulties with reasoning in mathematics, whatever the mathematical field studied. This question is well explored in the field of cognitive psychology (Richard, 1990), and also by didacticians (Radford 1985, El Faqih 1991, Duval 1995). More often, the logic system that is used for analysing these difficulties is propositional logic, truth-value system, even when the authors assume (as did Russel 1903) that in mathematics we need predicate calculus. According to us, three reasons at least may be given for explaining this matter of fact. The first reason is that, in France, reasoning abilities are developed mainly through geometry, for pupils 13-15 years old. For this purpose, the only syllogism taught is Modus Ponens : « *if p, then q; p; hence q* »³. The second reason is that, most often, teachers, as do mathematicians, don't explicit the quantification, specially concerning the conditional statements (Durand-Guerrier, 1996). The third reason is that when mathematic teachers introduce the logic language for formalizing mathematical statements, as it is done for postgraduate students, specially for calculus, they generally consider that it's enough to give some syntactic rules allowing a right use of symbolic formulae.

In our own investigations, we have shown clearly that the accurate logic system for analyzing the difficulties in mathematical reasoning is the predicate logic, even in geometry, and more specially the elementary model theory with Tarski's semantic conception of truth as presented in Tarski (1944), and developed in Quine (1950, 1960). Indeed, this allows taking care of the kind of mathematical objects you are

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³ See Duval., 1995.

working with , to explicit quantification and consequently the scope of the quantification. More over, in such a theory, you can also consider how the knowledge of pupils or students may modify their reasoning. As it is shown, we assume, according with Costa (1997), that to study logical-mathematical fields, it's necessary to grasp simultaneously syntactic, semantic and pragmatic aspects, with Morris (1938)'s acceptance for these three terms.

In this paper, we will illustrate these propositions through three examples held in various situations. The first one concerns the solution of an apparent contradiction in a pupil answer. The second one explores a disagreement between teachers and pupils about the truth value of a conditional statement. In the third one, we wonder about the possibly didactical obstacle created by using a non-valid logical rule in a proof held in a university level mathematical handbook for students.

I. General theoretical framework : about syntax, semantic and pragmatic

As the three terms syntax, semantic and pragmatic are widely polysemic, we intend to precise here the framework we use. We assume a logical point of view and follow the definitions as given by Morris (1938), definitions which are used by most authors working in formal semantic.

I.1. The syntax concerns the way the sentences or the formulae of a given language are built ; it follows rules (i.e. constraints). For example, the following formula “ $A \cap (B \subset C) = A \cap B$ ” (actually proposed by students) violates a syntax rule of set theory. Indeed, “ \cap ” is an operator that accepts two terms and provides a term, while “ $B \subset C$ ” is a binary relation (a predicate) that accepts terms and provides a proposition, or a “open sentence⁴”. As this formula appears while formalizing the sentence “*The intersection of a set A with a set B included in a set C is the same as the intersection of this set A with this set B*”, it illustrates the fact that the translation from “ordinary language” in a formal language don't respect necessarily the syntax and therefore needs that we take care of the logical status of the letters we use.

I.2. The semantic concerns truth values and hence references. According with Tarski (1944) and Quine (1950, 1960) the basic notions are : “open sentences”, “designation”, and “satisfaction for an open sentence by an assignment in a structure”. A structure Σ consists in a domain for objects (for example the integer numbers set N), function (for example *successor*, *addition*) properties (one place predicate, for example *to be primary*) and relations (two or more places predicate, for example *to be less than*) ; the syntax of the language provides sentences ; some of them are open (*see note number 3*). An open sentence F with n free variables x_1, x_2, \dots, x_n is satisfied by n objects a_1, a_2, \dots, a_n if the proposition obtained while assigning the object a_i to the variable x_i for every i from 1 to n is true in the considered structure. If every n -uplet satisfies the open sentence in a structure Σ , then Σ is a model for F . As said Tarski, this is exactly what we do in mathematics with equations.

Here is an example. Let us consider the following structure $\Sigma : \langle N, +, \times, 0; 1; s, \alpha, \beta \rangle$ with $+$ for addition, \times for multiplication, s for function, successor, α for even number, and β for primary number, and the open sentence F : “if x is an even

⁴ An open sentence has no truth value ; it appears when there are free variables ; in this case, « $B \subset C$ » is a proposition if B and C are two set already introduced ; if not, it is an open sentence ?

number, then its successor is a primary number.” 4 satisfies F ; indeed 4 is even, and 5 is primary ; 8 doesn't satisfy F , for 8 is even, and 9 is not primary ; notice that, although this might not be obvious for most of us, every odd number satisfies F ; indeed, the antecedent of the conditional is false.⁵ Then, we can close F in order to get a proposition in two manners : (1) “ $\forall xFx$ ” (for every x , Fx) and (2) “ $\exists xFx$ ” (for at least one $x Fx$). In Σ (1) is false and (2) is true. In Σ_1 with only integers from 1 to 7, (1) and (2) are both true ; in Σ_2 as $\{8 ; 14 ; 20\}$, (1) and (2) are both false. So the semantic as developed in model-theoretical point of view takes care of the objects you are working with, and the domain you consider. This leads to consider pragmatic as defined below.

I.3. The pragmatic concerns the context, the situation, the persons who are involved in the situation, and hence their knowledge about the situation. Consequently, the pragmatic aspect is more than referential function ; it involves not only the “real world”, but also what is possible and exploration of possibilities' field (Vignaux, 1976, p.273). We might thought that pragmatic doesn't concern mathematics at all, but our purpose is on mathematical education, and every teacher knows that neither the way a situation is understood in mathematical classroom, nor the truth value of sentences are necessarily the expected ones.

Let us consider a rather common classroom situation. The teacher says : “*let us consider a quadrilateral which diagonals are perpendicular : is it a rhomb ?*”. Here are some possible pragmatic aspects: we are at primary school/ middle school/ high school/ university; the pupils (students) have already/never met counter-examples, they have studied/not studied the theorems concerning diagonals; the quadrilateral is drawn/not drawn, and if yes pupils can/cannot see it; pupils are allowed/not allowed to draw, it is an exam, an evaluation, a problem session, they work alone/in collaborative groups: there is a debate etc.... Of course, you can recognize among this most of things studied in didactic in various theories. In our own work, we assume that a logical point of view enriched the didactical analysis of pupils reasoning, argumentation and more generally discourse. That's what we try to show now.

II. False or both true and false ?

How to solve an apparent contradiction in a pupil's answer ? Imagine a didactic situation in which a pupil seems to be assuming « p and non p » (syntactic point of view); is he (or she) illogical⁶.

II.1. About tertium non datur

Most often, teachers assume, as a law, that in Mathematics, every sentence is either true, or false. This rule is generally identified with *tertium non datur* principle; yet this is not exactly *tertium non datur*. In predicate logic, " $p(x)$ or non- $p(x)$ ", where p is a predicate, is a statement true in any model (*a logically valid statement, a tautology*), corresponding to *tertium non datur* principle, although neither " $p(x)$ " nor

⁵ For developpement about these questions, see Durand-Guerrier 1996

⁶ Notice that this question is not so strange as it looks like. In July 1996, a Symposium about *Teaching logic and reasoning in an illogical world* was held, sponsored by the DIMACS Special Year on Logic and Algorithms and the Association for Symbolic Logic in conjunction with the Federated Logic Conference. Hosted by Rutgers, State University, New Jersey. <http://www.cs.cornell.edu/Info/People/gries/symposium/symp.htm>

"non- $p(x)$ " can receive a truth value. Aristotle, already, distinguished between the two principles : the first one characterizes propositions, the second one can be applied to statements without truth value, and more over, you can assume *tertium non datur* even when you don't know which sentence, among " p " and "non- p ", is true (except, in certain cases, if you are intuitionist). As we said before, open statements do not have truth value. An important activity for mathematicians is to determinate for an open statement which objects satisfy it, and which do not. According to Lakatos (1976), looking for conjecture's counter-examples is very important for mathematics discovery.

II.2. Is n^2-n+11 a primary number for every n ?

In Arzac & ali (1989), which proposes mathematical situations for learning deductive reasoning for 12-13 years old children, we can find a situation dedicated to the rule "an example that satisfies a statement is not sufficient to conclude that this statement is true". The problem submitted to the pupils is to know if "for every n , n^2-n+11 is a primary number" is a true sentence or not. Pupils work first alone, then in small groups ; each group writes down a poster ; the posters are then collectively commented and there is a debate about the answers' validity. Relating the situation, the authors included a fragment of the dialog between pupils concerning the truth value of the sentence. On the poster that is discussed, it is written that the sentence is true ; there are some examples; other pupils have found the obvious counter-example 11 ; so they argue that as the sentence is not always true, so it is false, which is the expected answer. However, some pupils, G. and M., don't want to declare that the sentence is false; there are several examples (at least every integer from one to ten), and at the moment, only one counter-example ; later a pupil gives 22 and 33 ; but M. is not yet convinced for "they are all multiples" ; it's only when 25 appears as a counter-example that M. gives up. As for G., she says that it is true, and false. So G., less or more, seems to assume that a sentence might be both true and false, which might be considered as illogical for this violates the contradiction principle. According with Quine (1960), we prefer interpret it as a linguistic disagreement.

Instead of considering that G. assumes " $p \wedge \neg p$ " (syntactic point of view), we may understand that she means « there is a that satisfy " p ", and their is b that satisfy "non p " », in other terms, " $\exists aP(a) \wedge \exists b\neg p(b)$ " ; (semantic point of view). This offers a way to solve the contradiction. As a theoretical position, we think that we must follow the Charity principle as defined by Quine and Davidson⁷, considering that the fact that a pupil is illogical is less probable than a misunderstanding. On an other hand, we can see here that the teacher insists on the fact that a sentence with a counter-example is false (In mathematic, it's like this !) ; however, M. who tries to "save" the truth of the sentence by reducing the domain considered is not so far from mathematical practice ; indeed, it is quite relevant to look the truth value of the sentence in a structure which domain is N minus all the 11's multiples. According with this point of view, it is possible to change this kind of situation, proposing open sentences, and asking for the largest domain on which the sentence is true; in this case, pupils can't give a definitive answer, because they can't characterize examples

⁷ For a presentation, see for example Delpla (2001)

and counter-examples ; but in other cases, such a question may lead to elaborate one, or two, or more theorems. The difference we can see here between children's point of view and teacher's one emphasizes the difficulties with conditionals theorems that are not bi-conditionals. In that case, teachers say that the converse theorem is false; yet usually, the converse open statement has many examples, and even advanced students do not agree with saying it is false. Then they do not recognize the lack of inference and may assume invalid deductions. This pleads for investigating, in classroom, about models for open sentences, beyond the necessary search of counter-examples.

III. True, false or can't tell ?

How can we understand that some good pupils declare, concerning a conditional statement, that « they cannot decide if it is false or true », while teachers think that it's obviously false ?

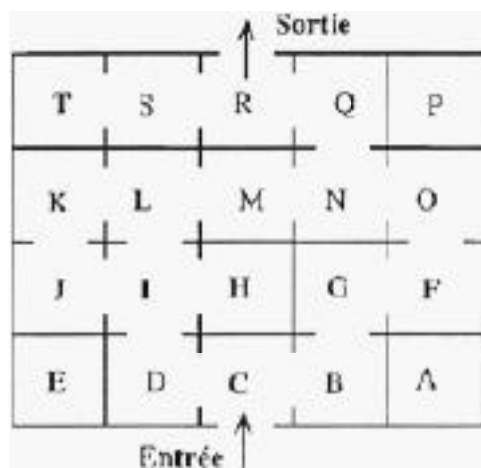
III.1. Contingent statement for a subject at a certain moment

There is, in predicate logic, a rule named " universal instantiation ". When "for all x $F(x)$ ", where F is a sentence with exactly one variable non-quantified, is true in a certain set, then for every element a of this set, we may infer " $F(a)$ ". According to this rule, we get an action rule for a subject solving a problem: as soon as a subject knows "for all x $P(x)$ " is true in a certain domain, he may infer $P(a)$ for every element a of the domain. More precisely, he can tell that, necessary " $P(a)$ " is true. On the contrary, when the subject knows that "exists x $P(x)$ " is false, he can infer for every element a of the set, that " $P(a)$ " is false . On the other hand, when "for all x $P(x)$ " is a false sentence and "exists x $P(x)$ " a true sentence, it is possible that " $P(a)$ " is true, and it is possible that " $P(a)$ " is false. In that case, " $P(a)$ ", which has a truth value, is contingent for the subject as far as he is able to know the truth value of the sentence. So, for a subject solving a problem, at certain steps of his search, some sentences may be necessarily true, impossibly true (necessarily false) or contingent (possibly true, possibly false) according as he knows, or not, a convenient general theorem. We can illustrate this with an example abstracted from an evaluation concerning 15-16 years old pupils.

III.2. The labyrinth task

This task is submitted to pupils 15-16, in mathematics class; it's an evaluation elaborated by teachers involved in didactic search⁸ and proposed by voluntary teachers to their own pupils. Subjects are told that a person named X managed to cross a labyrinth and never use twice the same door. The labyrinth is drawn. There are twenty rooms on four levels pointed by letters A, B, C, \dots to T . Three ones have no door: A, B & P . Two have exactly one door: H & T . Three ones have three doors: L, N & R ; one has four doors: I . The other ones have exactly two doors. According with the configuration, you necessarily enter the labyrinth in room C and leave it crossing successively N, Q, R . (see figure above)

⁸ EVAPM2/91, Association des Professeurs de Mathématiques de l'enseignement Public (APMEP, France)



The authors write:

“We may state sentences relevant to the situation. For some of these sentences, we can state a truth value (TRUE or FALSE); for others, we don't have enough information to decide if they are true or not; (in that case, answer CAN'T TELL). For example, the sentence " X crossed C " is a true sentence. Indeed, we affirm that X crossed the labyrinth, and C is the only entrance room. “

Then they propose the six following sentences: 1- X crossed P ; 2- X crossed N ; 3- X crossed M ; 4- If X crossed O, then X crossed F ; 5- If X crossed K, then X crossed L ; 6- If X crossed L, then X crossed K.

Sentence one is necessarily false; indeed, P has no door. Sentence two is necessarily true as we said before. Sentence three has a truth value; but we can't know it without further information ; the right answer is "can't tell". Sentence four is necessarily true; indeed O is a room with exactly two doors and one is common with F; Sentence five is necessarily true for a similar reason. For sentence six, we can't know the truth value; indeed, you can cross the labyrinth, crossing successively C,D,I,L,M,N,Q,R; in that case the sentence 6 is false; but you can also cross it, crossing successively C,D,I,J,K,L,M,N,Q,R, and in that case the sentence is true ; so, the right answer is "can't tell". According to the authors, most of pupils (60%) answered " can't tell " for sentence 6; the surprise comes from the teachers themselves who consider that this answer is wrong! They give as an example of false reasoning the following argues:

“The sentence number six is neither true nor false. We can't tell. For X might crossed through K, but might also cross through I, a room which has a common door with L, avoiding so K. “

Except for the fact that the sentence number six has actually a truth value, we agree with this answer. However, we can understand the teacher's point of view through this notice concerning the conditionals sentences number 4 to 6:

"Are they mathematical statements, which we must understand in their whole? In that case, the important matter is the bound between the two sentences and not the particular truth value of each one."

So, for the authors, the conditional statement is clearly the Russell's generalized conditional, and in the sentences number four to six, X is a universally quantified

variable, which is not the case in the sentence three for which they expect the answer "can't tell". In fact, although the person is named X , X is not here a variable; we might have call her Paul or John or every else. More over, there is no referee population; endless, to describe the situation in logical language, the relevant variable is the "crossing", as it appears in spontaneous treatment. Doing this (a crossing is a succession of letters among the letter from A to T , with some rules), we can see that for sentences three and six, the formal open sentences corresponding lead to a false universal sentence and a true existential sentence; so, the formalization of the task allows us to make clear that point : the truth values of sentences three and six are not constrained by the situation.

The teachers' point of view corresponds to a very common practice in mathematics classes, in France. Indeed, it is nearly never assume that some sentences may be contingent for the subject. However, this experiment and others (see Noveck 1991, p 95) shows that when "can't tell"'s choice is given, pupils use it. So, in a certain way, implicit quantification in mathematics class prevents the emergence of contingent statements, which are rather "natural" for pupils and students.

IV. Valid or not valid ?

How can we decide if it is valid or non valid to use the following rule :

“ For every a , their is b such as fab and for every a there is b such as gab , so for every a there is b such as fab and gab ” (R_1) ?

Imagine a calculus course⁹ in which this rule is implicitly assumed for demonstrating that "if f and g have h and k respectively for limit in c , then $f+g$ has $h+k$ for limit in c ". Probably, the proof will be considered as a correct one by most of mathematicians ; but how can a student, just beginning studying calculus, know when this rule might be used, and when it must not be used ?

IV.1. Natural deduction in predicate logic, a tool to control proofs

For the question of rule R_1 's validity, an answer can be given through mathematics, if you know enough mathematics, as we will see below . It can also be given by logic, and especially through natural deduction for predicate logic (Copi, 1954, Quine 1950). Indeed, this system provides us logical proofs in order to demonstrate the theorem of the predicate calculus, the logically valid statements, merely named valid statements. For this, it gives four rules for introducing and eliminating universal and existential quantifiers, and some restrictions about the use of letters introduced by eliminating existential quantifiers: such a letter must not be used for a new elimination, and can't be involved in introduction of universal quantifiers. A main interest of this system, compared with other ones, is that it holds rather near with classical mathematical proofs. More other, we can use this system for controlling mathematical proofs, specially proofs by « generic element » (we prove « fa » for any a , so we have proved « for every x , fx » (corresponding with the rule named « universal generalization »).

⁹ Houzel, C. (1996)

IV.2 Where using the rule R_1 leads to an incorrect mathematical proof

Many students meet strong difficulties when studying calculus, especially when they have to deal with statements involving two different quantifications, such as “ $\forall x \exists y Fxy$ ”. It is obvious that, as soon as you have to prove theorems, the frame proposed by Duval for geometry is not relevant. The incorrect following proof will illustrate this point. We first recall a well-known theorem

Theorem 1. Let us consider two real numbers a et b such as $a < b$ and a function f defined on bounded interval $[a; b]$. If f is continue on $[a; b]$, and derivable on open interval $]a; b[$, then there is a real number c in $]a; b[$ such as $f(b) - f(a) = (b - a)f'(c)$.

The theorem to prove is a generalisation of the previous one with two functions

Theorem 2. Let us consider two real numbers a et b such as $a < b$ and two functions f and g defined on bounded interval $[a; b]$. If f and g are continue on $[a; b]$, and derivable on $]a; b[$, and if the derived function g' is never equal to zero on $]a; b[$, then there is a real number c in $]a; b[$ such as $\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$.

A proof rather often provided by students in first year scientific university consists in a deduction from theorem 1 toward theorem 2 as below :

Function f satisfies the conditions for applying theorem 1; hence there is a number c in $]a; b[$, such as $f'(c)(b - a) = f(b) - f(a)$. Also g satisfies the conditions for applying theorem 1; hence there is a number c in $]a; b[$, such as $g'(c)(b - a) = g(b) - g(a)$. As g' is never equal to zero on $]a; b[$, $g'(c) \neq 0$ hence $g(b) - g(a) \neq 0$. The result comes from the quotient of the two above equalities.

This proof is not correct ; it may be shown on an example, considering two functions such as it's not possible to choose “the same number c ”¹⁰. Analyzing this proof with Duval's frame shows that the involved mathematical theorems, explicit or implicit, are used in a right way. Incorrectness is not a consequence of a wrong application of modus ponens. The error in the proof can be analyzed with two different points of view. In the first one, we can argue that when we apply in a proof a sentence such as “ $\forall x \exists y Fxy$ ”, it's necessary to add that “ y depends on x ” ; so; when you apply successively two such statements, you have to change the letter in the second statement (you write for example “ $\forall x \exists z Gxz$ ”)¹¹. We will say that in this case *we study empirically the rules effectively used in mathematical proofs*. It's empirical for there is no mathematical, nor logical relevance to change the name of a mute letter (a bounded variable) in a statement ; it is a rule for action, in order to prevent errors. In the second one, we interpret the proof in predicate logic, and we use natural deduction extended to predicate logic as a tool for controlling validity. The error is here to use a letter for a bounded variable as if it was a letter for an object. The semantic inference following the assertion of the existential statement doesn't appears, and then the restrictions about the object introduced with this type of inference are not applied. In this case, *we use a theoretical model to describe the practise above*.

¹⁰ For example x^2 and $\sin x$; notice that for two polynoms with degree under two, you can choose the same number.

¹¹ For more development see Arsac & Durand-Guerrier (2000)

Anyway, the proof is incorrect, and this example provides a structure in which the statement R_1 is false ; this proves that R_1 is not a theorem in predicate logic ; it is not valid¹². Opposite with the proof for sum's limit, it is quite obvious that no mathematician will considered that this proof is correct. Yet, it is the same logical rule that has been implicitly used for the two proofs. The difference is that, in the first case, we can easily built a number that holds for the two functions, while, as we told above, it's generally not possible in the second case. The incorrect use of R_1 can be found in many situations, even in situations where it leads students to "prove" a false statement. Here is a very important difference between an expert, and a novice. If you are an expert in a mathematical field, you know when it is *dangerous* to slack off the rigor requirement while a novice has to learn it, in the same time he learns mathematical knowledge, and this can't be done separately. However, to slack off rigor, you need at least to gasp what rigor is. So, in a didactical purpose, in order to promote a right understanding of what are mathematical proofs, we claim that it is necessary to introduce semantic considerations in learning mathematics at university, and to offer students tools for controlling the proofs they study and the proofs they build. We assume that natural deduction in predicate logic is well profiled for this purpose, allowing to control validity in a rather economical way (for other examples see Arsac & Durand-Guerrier 2000)

Conclusion

The examples presented here, added with other ones described elsewhere¹³, show that the model-theoretic point of view as developed by Tarski offers a general framework for analyzing mathematical proofs or reasoning, in addition with classical didactical theories. Coming back with our general theoretical framework, we think that the cases described here emphasize the necessity of considering the three aspects we introduced : syntax (*the linguistic form of R_1*), semantic (*the mathematical objects we work with*), pragmatic (*the situation, and the subject's knowledge about the mathematical field*). We might also have related our analyses with the formal semantic as developed in linguistic by Montague (1974), whose program was to apply model-theory to natural languages, and Kamp (1981), specially the Discourse Referent Theory (DRT), which main interest is to introduce conceptual rigor in an empirical domain where it is easy to be loosed (Corblin, 2002, p.2). As for us, another interest is that a model-theoretic approach for argumentation might plead for continuity between argumentation and proof opposite with the idea of a cognitive discontinuance¹⁴. This is a track we are investigating.

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¹² Obviously, it is very easy to build more elementary models in which R_1 is false.

¹³ Durand-Guerrier, 1996, 1999, 2000.

¹⁴ See Duval (1995)

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