WHAT KIND OF PROOF CAN BE CONSTRUCTED FOLLOWING AN ABDUCTIVE ARGUMENTATION?

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Abstract

My interests focus on the comparison between the abductive argumentation supporting a conjecture and the related proof. In particular, .the purpose of my research is to show the importance of a structural analysis between them (from an abductive argumentation to a deductive proof, from an abductive argumentation to an abductive "proof"). I propose the Toulmin's model as a tool which can be used to detect and to analyse some structural continuities and some structural gaps between an argumentation and the following proof. This analysis allows identifying in the passage from an abductive argumentation to a deductive proof, one of the possible trouble met by students in the construction of a proof.

Introduction

The purpose of this paper is to analyse some aspects of the relationships between argumentation and proof in geometry. In particular, my interests focus on the argumentation supporting a conjecture and the consequent proof.

Starting from the Italian research concerning the *cognitive unity* between the two processes (Boero, Garuti, Mariotti, 1996), I developed a theoretical framework in order to analyse and to compare argumentation and proof from a structural point of view (Pedemonte 2002). Indeed, I think that the comparison between an argumentation and the consequent proof may be carried out considering their contents (some words, some expressions, the theoretical framework if it exists in the argumentation, and so on), and/or their structure (abductive, inductive, deductive and so on). In this paper, I present a part of this theoretical framework: a structural analysis between argumentation and proof considering an abductive argumentation[1].

In the student's protocols, it is easy to find abductive argumentations supporting a conjecture. But according to curricula of the secondary school, when students learn to proof, they usually have to construct deductive proofs. As the analysis will show, this passage it is not always immediate for them.

In this paper, Toulmin's model is used as a methodological tool useful to compare argumentation and proof from a structural point of view. In particular I want to show the gap which can be observed between an abductive argumentation and a deductive proof. The aim of this analysis is to present this gap as a cognitive gap. It is to say that in the structural gap between argumentation and proof, it is possible to identify some difficulties met by students in the construction of a proof.

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[1] Abduction refers to an inference which starting from an observed fact and a given rule, led to a conclusion (Peirce, 1960, Polya, 1962). An abductive step can be represented in the following way:

B

$\underline{A \Rightarrow B}$

B

B is an observed fact, $A \Rightarrow B$ is a rule. Then A is more probable.

The experimental design was carried out in some 12th -grade classes, when students begin to learn proof. I proposed a geometrical problem requiring the production of conjectures and the related proof. The students' productions were analysed according to Toulmin's model in order to highlight and to understand the cognitive relation between argumentation and proof.

Theoretical framework

The relationships between the production of a conjecture and the construction of proof has been an object of study from a cognitive perspective. Actually, research studies showed the possibility that some kinds of continuity exist between the two processes. In particular, continuity can take the following shape:

"*During the production of the conjecture, the student progressively works out his/her statement through an intensive argumentative activity functionally intermingled with the justification of the plausibility of his/her choices. During the subsequent statement-proving stage, the student links up with this process in a coherent way, organizing some of previously produced arguments according to a logical chain*" (Boero, Garuti, Mariotti, 1996).

This phenomenon is referred to by the authors as *cognitive unity*.

During the solving process, which leads to a theorem, we may suppose that an argumentation activity is developed in order to produce a conjecture. When the statement expressing the conjecture is made valid in a mathematical theory, we can say that a proof is produced. This proof is a particular argumentation based on a mathematical theory. The *cognitive unity*, described above, concerns the content between this argumentation and the consequent proof. During the production of several theorems, there are many similar content elements in the argumentation and proof, therefore we can say that it is frequent to find a continuity in the content of the two processes (Pedemonte, 2002).

On the other side, according to Duval (1991), deductive thinking does not work like argumentation: there is a "gap" between the two processes even if they use very similar linguistic forms and proposition's connectives. The structure of a proof may be described by a ternary diagram: data, claim, and inference rules (axioms, theorems, or definitions). Within proofs, the steps are connected by a "recycling process" (Duval, 1992–1993): the conclusion of a step serves as input condition to the next step. On the contrary, in argumentation, inferences are based on the contents of the statement. In other words the connection between two propositions is an intrinsic connection (Duval, 1992–1993): the statement is considered and re-interpreted from different points of view. For these reasons (according to Duval) the distance between proof and argumentation is not only logic but is also cognitive: in a proof, the epistemic value[2] depends on the theoretical status whereas in argumentation it depends on the content. Then it is easy to observe the cognitive distance between the two processes.

[2] The epistemic value is the degree of certitude or conviction associated with a proposition (Duval,1991).

The opposite positions of these researches moved me to search an answer to the following question: Is there cognitive unity or cognitive distance between argumentation and proof?

The experimentation that I carried out (Pedemonte, 2002), push me to distinguish the content and the structure in the argumentation and in the consequent proof. When student produces an abductive argumentation, a structural gap is necessary to construct a deductive proof. This gap can be a difficult for him. My hypothesis is that the "natural" continuity that appears in the contents of argumentation and proof, (in according with the hypothesis of cognitive unity) can be observed in their structure too. Indeed, sometime student produce an abductive "proof" after an abductive argumentation (see example 2, p. 6).

In order to give an answer to my question and to validate or invalidate this hypothesis I needed a tool to compare the argumentation structure and proof structure: the Toulmin's model.

Toulmin's model: a methodological tool

Toulmin proposes a model describing the structure of the argumentation (1958). I use this model as a tool to compare the structures relating to the two processes: argumentation and proof.

In any argumentation the first step is expressed by a standpoint (an assertion, an opinion). In Toulmin's terminology the standpoint is called the claim. The second step consists of the production of data supporting it. It is important to provide the justification or warrant for using the data concerned as support for the data-claim relationships. The warrant can be expressed as a principle, a rule and the like. The warrant acts as a bridge between the data and the claim. This is the base structure of argumentation, but auxiliary elements may be necessary to describe an argumentation. Toulmin describes three of them: the qualifier, the rebuttal and the backing. The force of the warrant would be weakened if there were exceptions to the rule, in that case conditions of exceptions or rebuttal should be inserted. The claim must then be weakened by means of a qualifier. A backing is required if the authority of the warrant is not accepted straight away.

Then, Toulmin's model of argumentation contains six related elements as showed in the following figure.

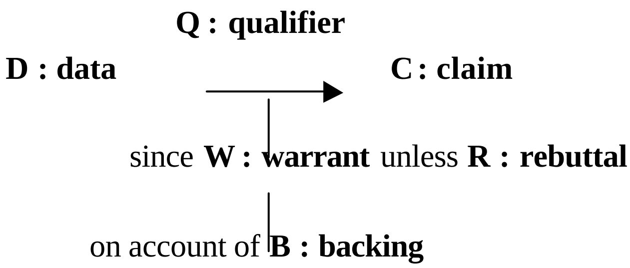


Fig. 1. Toulmin's model of argumentation[3].

[3] Let us illustrate this model with the same example used by Toulmin (1958) : *Claim :* Harry is a British subject; *Data :* Harry was born in Bermuda; *Warrant :* A man born in Bermuda will generally be a British subject; *Rebuttal :* No, but it

By means of this model, in opposite position respect to Duval, the argumentation structure has a ternary structure. Then the comparison between an argumentation and the proof subsequently produced is possible also from a structural point of view. If we consider a proof as a particular argumentation, the warrant is an axiom, or a definition, or a theorem, in a specific theory.

In the following section, I illustrate the use of this model as an example when the resolution process of an open-ended problem will be analysed.

Experimental design

The following examples are taken from a set of data collected in four 12th -grade classes in Italy, and in one 12th -grade class in France. The students worked in pairs on a computer running the Cabri-Geometry software. The experiment lasted an hour and a half. The problem proposed was the following:

Problem. ABC is a triangle. Three exteriors squares are constructed on the triangle's sides. The free points of the squares are connected defining three other triangles. Compare the areas of these triangles with the area of triangle ABC (see figure p. 5).

I will transcribe a part of two solution protocols related to the proposed problem. Theses parts are based on the transcriptions of the audio recordings and the written productions of the students.

I will present two protocol examples:

- Example 1: Example of structural gap between argumentation and proof (abductive argumentation into deductive proof
- Example 2: Example of structural continuity between argumentation and proof (abductive argumentation into abductive "proof").

In order to analyse the argumentation, I have selected the assertions produced by students and reconstructed the structure of the argumentative step: claim C, data D and warrant W. The indices identify each argumentative step. The student's text is in the left column, and my comments and analyses are reported in the right column. The texts have been translated from Italian into English.

Example 1

The analysis starts at claim C_7; at this point students are comparing the area of the triangle ABC and the area of the triangle ICD. Till now the students spoke about the construction of the heights of the two triangles. They decided to construct the heights in order to compare the areas of the triangles ABC and ICD.

…. Students construct the heights of the triangles ABC et ICD 31. L: I'm prolonging the straight line, yes, the straight line on the segment… what have I done?	*The figure as represented from the students using Cabri-géomètre.*

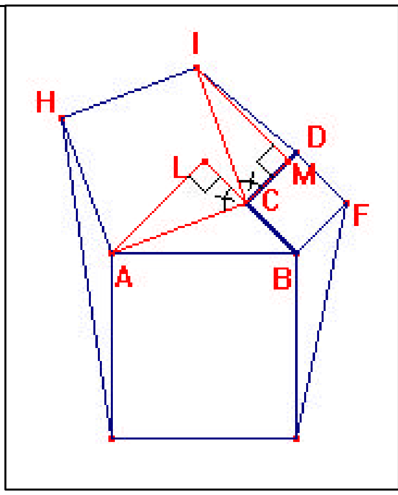
generally is. If his parents are foreigners or if he has become a naturalised American, then the rule doesn't apply; *Qualifier :* True : its only presumably so; *Backing :* It's embodied in the following legislation :…

32. G: The straight line by the points B and C
33. L: ah it's true !
34. G: now, we need to do the line perpendicular to this line
35. L: ah there that's it done but you know that it seems they are equal…
36. G: almost equal !
37. L: not anymore, it seems that they are perpendiculars, I have observed this before

…….

44. Students together: hey, these are two equal triangles !
45. L: it's true, ALC and ICM these are two equal triangles…what do they have?
46. G: we realized… then AC is equal to IC because they are sides of the same square
47. L: wait!
48. G: AC is equal to IC because they are sides the same square, after
49. L: LC…
50. G: it's equal to CM, why ?
51. L: Then… Because it's equal to CM… in my opinion, it's better to prove … no wait this angle is right and this angle is right too.

……



C_7: The heights seem to be equal.
C_8: The heights seem to be perpendiculars.

The statements are "facts" where the epistemic value is joined to perception of the figure in Cabri-Geometry.
The Cabri-Geometry drag allows them to see the small triangles. The students realize that the heights are the heights of two equal triangles. The statement is now a fact.

C_9: The triangles ALC and ICM are equal.

The structure of the argumentative step is an abduction:

D_9= ? ——→ C_9

W: congruence criterion

The structure of the argumentation is that of an abduction. The students see that the small triangles constructed on the height (ALC and ICM) are equal and they search for a theorem to prove this fact. During the proof, students make data D_9 explicit in order to affirm that triangles ALC and ICM are equal. The abductive structure of the argumentation is transformed into a deductive structure in the proof. Once obtained, claim C_9 is used to deduce that the heights of the triangles ABC and ICD are equal and consequently that theirs areas are equal.

The students write the proof:

I consider the triangle ABC and the triangle ICD. At once I consider the triangles ALC et ICM and I prove that they are equal triangles for the SAA congruence criterion because we have:

- AC = IC because they are two sides of the same square
- ALC = IMC because they are right angles

The proof structure is a deduction:

D_9: AC =IC
ALC = IMC
ACL = ICM
——→ C_9: the triangles ALC and ICM are equal

W: SAA congruence criterion

If the triangles are equal then it's possible to conclude that the heights are equal, and finally

<table>
<tr>
<td>(angles constructed as intersection between the sides and the heights)
• ACL = ICM because they are complementary of the same right angle (- LCI)
In particular IM = AL. Then the triangles ABC and ICD have the same base lengths (as sides of the same square) and the same heights, then they have the same area.</td>
<td>then the areas are equal because the bases are equal.
The conclusion C_9 of the previous step is the date D_{10} to apply the inference to the second step.
D_{10}: C_9 → C_{10}: the heights are equal
W: inheritance
D_{11}: C_{10} → C_{11}: the areas of the triangles ABC and ICD are equal
W: formula of area</td>
</tr>
</table>

The protocol appears to be an example of *cognitive unity*. Indeed, students use the "SAA congruence criterion" both in the argumentation and proof in order to justify the statements. Words and expressions used in the two processes are often the same ("triangles ALC and ICM are equal", "heights are equal", and the like). But looking more carefully, we can observe a gap between the structures of the two processes: we find an abductive structure in the argumentation (from D_9 to C_9) that is transformed into a deductive structure in the proof.

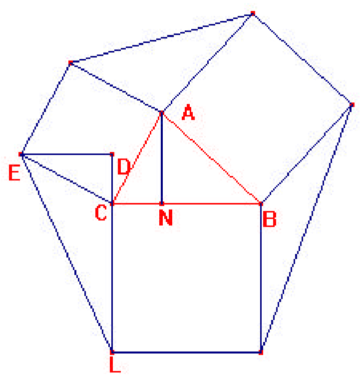
In this case it seems that students don't have met difficult in the passage from an abductive argumentation to a deductive proof. On the contrary, in other examples, as the following, the presence of some abductive steps in the proof can be considered as a difficult that students meet when they construct a proof.

Exemple 2

The analysis starts at claim C_6; at this point students affirm that the area of the triangle ABC and the area of the triangle ECL are equal. They have calculated these areas. The statement C_6 is the statement conjecture that students have to justify.

<table>
<tr>
<td>Students calculate the areas of the triangles

103. C : The areas are always equal
… with the calculator the areas are equal
104. N : now we have to proof it!
105. C : we need to find how the base and the height change …if there is a rapport that takes the area constant

106. N : we need to find a connection with the interior triangle … changing the triangle the areas are constants</td>
<td>The figure as represented from the students using Cabri-géomètre


C_6: The triangles' areas are equals
This statement is a fact.</td>
</tr>
</table>

107. C : the area is constant… but I don't understand…then we have to find base for height equal to base for height of the other triangle 108. N : if we take the bases constant and we change the heights.	*In order to justify this fact, students look for a rapport between bases and heights, which takes a constant area. This step of argumentation is an abductive step.* **? D_6 : rapport for a constant area** → **C_6 : The triangle's areas are equal** W: Formula of the area *If this rapport has to be constant, the heights have to be equal because the bases are equal. The argumentation's step is an abduction.* **D_7 : equals bases ? equals heights** → **C_7 : D_6** W: transitivity of equality
115. C :But why the heights are equal? 116. N :we have…. We have the same base and ….yes, it's true we have to proof that they have the same height… but we have that this side is equal to this side of the triangle ABC … 117. C : then the little triangle is equal to the other little triangle … 118. N : wait wait… yes it's true two sides are equal… then 119. C : then there is an angle of 90° 120. N : we need another side or another angle… for example this angle is equal to this angle because. .. ….	*Students have to justify why heights are equal. For this reason they consider the little triangles DANC et DDEC constructed on the heights. If they are equal, their heights are equal.* **? D_8 : two little triangles are equal** → **C_8 : D_7** W: inheritance *The equality between the two little triangles can be justified by the congruence criterion. It's necessary to find data in order to apply this theorem. The step of argumentation is also an abduction.* **? D_9** → **C_9 : D_8** W: SAA congruence criterion *As in the previously example, students look for data to apply congruence criterion and justify the equality between the two little triangles.*

The argumentation structure is an abduction. Students are conscious that the areas of the two triangles are equal (they calculate these areas). Then they look for a rapport between bases and heights that take the area constant (step 6). The comparison between heights drives explicitly here, to a comparison between the two little triangles ANC et EDC constructed on the heights. For this reason students look for data to apply one of the congruence criterion (step 8 and step 9). In the proof constructed by the students there is an abductive step.

Students consider the triangles ABC et ELC	*Structure of proof contains an abductive step.*
We know that this base is equal to the base of the triangle. Now we have to proof that the heights are equal. We have verified this fact by means of congruence criterion proved on the sheet with drawing. *On the sheet with drawing:* Triangle ANC = Triangle EDC EC=AC EDC=ANC=90° ACN=ECD because ACE=90°, DCN=90° and if the angle DCA is removed from two other angles we have the same angle.	**D_6 : equal bases** **? equal heights** → **C_6 : The areas of triangles are equal** W: formula of area **D_8 : DANC=DEDC** → **C_8: equal heights** W: inheritance of equality **D_9 : EC=AC** **<EDC=<ANC=90°** **<ACN=<ECD** → **C_9 : DANC=DEDC** W: SAA congruence criterion

The redaction of this proof describes the abductive reasoning made by the students. A structural gap seems to be necessary to construct a deductive proof. But these students didn't arrive to cover this gap. The argument 6 is still an abductive step. This is the reason why in this case we can observe a structural continuity between argumentation and proof.

In the protocols that I analysed during my research (Pedemonte 2002), frequently I found structural continuity between an argumentation and the proof subsequently produced. This kind of continuity doesn't help students to construct a deductive proof. On the contrary, it is in this natural continuity that we can localize some difficult that students met in the construction of a proof.

Conclusion

In this paper, I presented a part of a theoretical framework constructed to analyse an argumentation and proof subsequently produced from a structural point of view. By means of Toulmin's model, we have observed a possible gap between an abductive argumentation and a deductive proof and we have observed also a possible continuity between the two structures.

We cannot undervalue the importance of the structure in the comparison between argumentation and proof; it is not unusual that the student tries to transform abduction into deduction during a resolution process, sometimes successfully, sometimes without getting an acceptable solution. The last analysis (example 2) is a clear, and not unusual example of structural continuity between argumentation and proof. For this reason, in the "natural" structural continuity between an abductive argumentation and a proof we can perhaps find one of the possible difficulties met by students in the construction of proof.

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