AN EDUCATIONAL EXPERIMENTATION ON GOLDBACH'S CONJECTURE

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Summary

The aim of this paper is to investigate mental representations of pupils about Goldbach's conjecture to improve the mathematical education from a historical viewpoint. It is based essentially upon two hypotheses of search: the first one concerning pupils' inability to represent mentally any general method useful for a demonstration; the second one concerning their intuitive ability to recognize the validity of the conjecture. The verification or the invalidation of these hypotheses are very useful in order to understand metacognitive processes which are basic for the learning phase and the cultural growth of pupils.

Résumé

Le but de ce papier est étudier les répresentations mentales de les élèves sur la conjecture de Goldbach pour améliorer l'éducation mathématique par un point de vue historique. Il est basée essentielement sur deux hypothèses de recherche: la premier relatif à la leur incapacité de répresenter mentalement une méthode général pour une démonstration de la conjecture; la second relatif a la leur habilité intuitif sur la validité de la conjecture. La vérification ou l'invalidation de les hypotheses seront utiles pour comprendre les processes metacognitif qui sont fondamental pour l'apprentissage et la croissance culturel de les élèves.

Undoubtedly, the intersection of mathematics teaching and the history of mathematics has had a long, fruitful tradition; so, nowadays one of the most frequent questions is: in which way can the history of mathematics influence mathematics education? If it is true that most knowledge is a response to a question, it is as true that the history of mathematics shows that mathematical concepts are constructed, modified, and extended in order to solve problems, so an alternative way of writing a history of mathematics is that of a history of problem solving. The pedagogical value of open problems and conjectures for mathematics teaching is in general remarkable, expecially in the educational methodology of problem-solving.
In fact, such a methodology is important above all for many reasons, some of which are:

- it allows pupils to use their acquired knowledge to solve problems;
- it improves their logical-deductive abilities;

- it contributes in consolidating knowledge already mastered in a consistent fashion.

Thus the aim of this research was to analyze the educational value of mathematical conjectures to improve some pupil's abilities. By facing a conjecture a pupil may be stimulated in acquiring his own ways of reasoning by either following his particular mathematical background or individual intuitive approach in order to solve a question.

Really, there is, in general, a difference between an open problem and a conjecture: by a conjecture we mean a statement which is thought to be true by many, but has not been proven yet. By an open problem we mean a statement for which the evidence is not very convincing one way or the other. For example, it was conjectured for many years that Fermat's Last Theorem was true, while it is an open problem whether $4! + 1 = 5^2$, $5! + 1 = 11^2$, and $7! + 1 = 71^2$ are the only squares of the form $n! + 1$. However, there is often a thin border between a conjecture and an open problem. The relationship between argumentation and proof, strictly connected to the relantionship between conjecture and valid statement, has been recently analized (see, for example, Pedemonte, 2000); in the present case I wanted to analyze the gradual passage of pupils' attempts from an argumentation to a demonstration, while they were facing a known unsolved conjecture. I chose a historical conjecture like Goldbach's one essentially for the simplicity of its statement and its fascinating empirical evidence. It states that:

"Every even number greater than 2 can be represented as the sum of two primes."

The conjecture was first formulated in a letter, dated Moscow, June 7, 1742 from Goldbach to Euler but it was not published[1], however, until 1843.

Perhaps, by chance, perhaps by the passionate reading of Fermat's number theoretic notes, Goldbach had been attracted by the apparent regularity of the following partitions:

6 = 3 + 3 8 = 3 + 5 10 = 5 + 5 12 = 5 + 7…

Euler claimed to be convinced of the truth of the conjecture, and so all that he could do was adjust the proof, or a strategy for it. It seems Euler never attempted to demonstrate the conjecture, but in a letter in December 1752 he stated the further conjecture (probable suggested by Goldbach too) that every even number of the form 4n+2 was the sum of two prime numbers of the form 4m+1. So, for example: 14 = 1 +13; 22 = 5 +17; 30 =1 + 29 =13 +17.

George Cantor (1845-1918) in 1894, prepared a very interesting table by which he verified the conjecture up 1000.

Other many mathematicians based their attempts upon Cantor's approach. For example, from 1896 to 1903 A. Aubry also verified Goldbach's conjecture for numbers from 1002 to 2000, and always in 1896 R. Haussner verified it up to 10000, by prepearing a set of tables. But even if these tables were interesting because they

[1] The letter was published for the first time by P.H. Fuss, *Correspondance mathématique et physique de quelques cèlébres gèométres du XVIIIème siècle*, tome I, St. Pétersbourg, 1843. However, the correspondence between Euler and Goldbach has been quite published by A.P. Juskevich and E. Winter, *Leonhard Euler und Christian Goldbach*, Berlin, Akademie-Verlag, 1965.

furnished a fascinating empirical evidence for the truth of the conjecture, on the other hand these direct procedures were very undesirable and much too laborious in order to obtain extensive tables which could really suggest a way to demonstrate the conjecture. Finally, in 1966 Chinese mathematician Chen Jing Run (1933-1996) established that every large even number is the sum of a prime and a product of at most two primes.
Goldbach's conjecture seemed to be useful in order to point out some interesting points as the following ones:
- pupils' conceptions upon a conjecture faced during the historical development of mathematics;
- pupils' attempts proving a conjecture reclaimed from history and compared with their argumentative processes;
- pupils' abilities in carrying out non-standard solving strategies (lateral thinking).
As we know a conjecture can be transformed into a theorem if a demonstration justifying it is produced; namely, if it is possible to use a mathematical theory allowing the construction of a demonstration of it.

Summary of the experimental work

In the first phase of my experimentation, which was realized with 88 pupils attending the third and fourth year of study (16-17 years) of secondary school, I used the method of individual and matched activity. Pupils working individually were expected, within two hours, to answer the following question:

1. *Is it possible to express the given two even numbers like addition of two prime numbers (by one or more ways)? Use the given table of prime numbers.*

   248; 356; 1278; 3896;

2. *If you answered the previous question, can you show that it is valid for each even number?*

The pupils working in pairs were expected, within an hour, to answer this question (in a written form and only if they agreed):

1. *Is it correct that each even natural number greater than 2 is a result of addition of two prime numbers? Give the reason for your procedure.*

In both cases the procedure was acoustically recorded. The second phase of the experimentation was carried out in three levels: pupils from the first school (6-10 years), pupils from primary school (11-15 years) and pupils from secondary school. The experiment was carried out on the lowest level in two phases: In the first phase the pupils could answer this question:

How could we obtain the first 30 even numbers using the prime numbers from the table which has just been made?

In the second phase, the pupils created small groups and tried to answer the following question:

Can you set even numbers made just as addition of two prime numbers? If yes, can you say that it is true for an even number?

The pupils from primary school solved the following problem within 100 minutes:

Is the following statement always true? "It is always possible to divide an even number like addition of two prime numbers." Explain your argument.

The procedure had four phases:

a) discussion about the task in pairs (10 min.)
b) individual written description of a chosen solving strategy (30 min.)
c) dividing of the class into two groups discussing the task (30 min.)
d) proof of a strategic processing given by the competitive groups (30 min.)

Pupils from secondary school solved the same problem like the pupils from primary school in the same way and within the same time limit.

Individual works were analyzed (a-priori analysis), the identification of parameters was carried out and those were subsequently used as a basis for the characteristics of pupils' answers. It enabled to do a quantitative analysis of the answers, to establish by the software of inferential statistics CHIC 2000 (*Classification Hiérarchique Implicative et Cohésitive*) and the factorial statistical survey S.P.S.S. (*Statistical Package for Social Sciences*) an implicative graph (graph functionality), hierarchical diagram, diagram of similarities and also factor data analysis. As an example, here is some of the possible answers by pupils (a-priori analysis):

1) He/she verifies the conjecture by natural numbers taken at random.(N-random)
2) *Golbach's method 1*
   He/she considers odd prime numbers lesser than an even number, summing each of them with successive primes. (Gold1)
3) *Cantor's method*
   Given the even number $2n$, by subtracting from it the prime numbers $x = 2n$ one by one, by a table of primes one tempts if the obtained difference $2n - x$ is a prime. If it is, then $2n$ is a sum of two primes. (Cant)
4) *Euler*
   He/she is uneasy to prove the conjecture because one has to consider the additive properties of numbers. (Euler)
5) *Chen Jing-run's method (1966)*
   He/she expresses an even number as a sum of a prime and of a number which is the product of two primes. (Chen)
6) He/she does not argue anything for the second question. (Nulla)
7) He/she thinks the conjecture is a postulate. (Post)

Hypotheses of search

The whole experimentation based itself essentially on the following hypotheses of search, which could be either validated or falsificated:

I. *Pupils are not able to go beyond the empirical evidence of the conjecture because they do not know how to represent mentally any general method useful for a demonstration.*

II. *Pupils can reach only intuitive conclusions about the validity of Goldbach's conjecture.*

Some examples of protocols

R. and S. are the initials of two pupils' names. The pupils are attending the fifth year of study (18 years) of secondary school

R. Hm … one asks to demonstrate if every even natural number is a sum of two primes.

S. Certainly, the experimental verification will not yelds to anything, because there can exist always a number umproving it.

R. The even number must be greater than 2 … and … if is it a postulate?

S. In fact, 2 = 1+1 and 1 is not a prime!

R. Well, wait a moment … we can state the theorem: hypothesis: if we have two primes, x, y and an even number …

S. Wait a moment … we can't say: if we have two primes … we must say: if one has an even natural number …

R. We have to find a manner to represent a prime number …

S. A prime? But, are you sure? I think there is not a way of representing a prime …No, no … we can get an even number:

2n = n+n or n-1 + n +1 or n-2 + n + 2 …

R. In my opinion we have to find a way to represent a prime …

S. Again? No, no, I'm not sure one is able to represent a prime in general …

R. Well, if we write the sum of n-k and n+k …

S. But it's trivial! We only know that if the number is odd, then it is not a sum of two primes …

R. Hm … I'm not sure about it … look: 15=2+13; 25= … no …. Let me think … 25=2+23 …

S. Well … I was wrong … in my opinion … perhaps … why don't we attempt to find a counter-example?

R. Yes, it's a good idea … a counter-example … 2n = a + b … a, b not primes …

S. 46 = 3 + 43 … no … 52 = 3 + 5 + 7 + 37 … no, they have to be two primes … in my opinion we have to attempt to prove it in general ..,.

R. Yes, let us attempt; well, if n=2k is an even number …

S. No, we have to set the hypotheses well … let n=2k an even number …

At this point the dialogue ends and the pupils sketch the following proof:
Hp. x, y primes
a=2n
n>2, x, y, n natural numbers.
Th. a = x + y.

Proof: Since *a* is even it must be equal to the sum of two even numbers or two odd numbers.
First case: n=4 impossible for the hypothesis;
Second case: x =2k + q
y=2h + t (k, h, q, t natural numbers)
q, t are odd because they are the difference between an odd number and an even one:
x – 2k = q
y – 2h = t
We assume 2k and q coprimes, that is 2k+q is not factorizable; analogously for 2h and t; so, *a* is the sum of two prime numbers. c.v.d.
Authors's note: an even number can be intended as a sum of two primes one of which is fixed while the other is chosen among the other primes by a table of primes.
Example:
20=3+7
40=3+37 …….
Final note. The two proofs, the first one more general, the second one more specific, set some obligatory conditions, which can contradict the text, so the statement could be a postulate.

Here are two examples of an individual protocol by two pupils: the first one attending the third year and the second one the fifth year of secondary school.

1. Every even number, as we know, is the sum of two odd numbers; prime numbers are also odd numbers, apart the first one, because, on the contrary, they would be trivially divisible by 2 and they would not be prime numbers. So, we can state that the sum of two prime numbers (apart 2) gives not only an even number, but that the same number can be expressed by another pair of prime numbers, as 248 which is equal to 181+67, 241+7, 211+37 … and so on. This is obviously valid for all the numbers greater or equal to 10. So, [it follows] the fact that two prime numbers always form an even number, and therefore that an even number is always the sum of two prime numbers."

2. a) 151+97=248; 241+7=248; 349+7=356; 337+19=356; 1259+19=1278; 1249+29=1278; 1237+41=1278; 3877+19=3896; 3853+43=3896.

Commentary (by the pupil)

These are only some of possible combinations to obtain as a result an even number by the sum of two prime numbers.
Facing this problem, and having at my disposal a table of prime numbers up to 10.000, the first thing which is crossing my mind is to make some attempts at random to obtain the proposed result. This is easy to me with roughly little numbers, but it is more difficult when the numbers are large. First, I thought to solve it by a little expedient: let us examine, for example, the number 356. The last digit is 6. It can be determined by a sum of [the digits] 7 and 9 which gives as a result a number ending by 6. So, I think I can look for two primes numbers which, ending by 7 and 9, for example, make as a result 356. By trials I reach it more easily. I am not satisfied and I try another way. <u>By it I am inclined to look for a prime number close to the one sought and for one which summed to that one gives the sought result</u>.In fact, for example, if I sum "349+7" I have the result. My method consistes in reducing the number "349" (347-337) and in finding a little number (13-17-19) which summed to gives me the result. For me this is the easier method.
b) The fact that an even number can always be the sum of two prime numbers is something that can be taken for granted. Trying to demonstrate that it is a very difficult thing I do not know why my mind is thinking about the fifth euclidean postulate. Surely, or almost at 99%, studying practical examples is for my part impossible because one would try to work upon an infinite numerical field ... According to me, one would have to work on one rule, on one formola, by which to obtain practical examples.

Some conclusions

For lack of space I can only summarize the main conclusions of my work.
- First of all, the two initial interviews closed with a strong claim by both of the pairs, namely that Goldbach's conjecture was really a postulate, so an undemonstrable assertion; this is a strong conclusion because it implies that there is a misconception by pupils about the meaning of "postulate", and it should be advanced by further experimentations.
- Without an a-priori analysis based upon historical attempts by mathematicians throughout centuries it should not be possible to analyze profitably pupils' attempts.
- As for pupils' approach in solving Goldbach's conjecture, both experimentations showed that essentially most of them based on numerical evidence, and only some of

them extrapolated their results from a finite set of values to the infinite set of positive integers, but without showing how they passed from trial and error to the conviction of the general validity of Goldbach's conjecture. This is a delicate point which should be advanced by further investigations:
- As for the two hypotheses which my work based on I deduced that almost all of the trials of pupils come down to the historical attempt of Golbach. So, this validated the two initial hypotheses of search.
- Some of the pupils, were wrong because they exchanged the statement of the conjecture by the converse, which is trivial.
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Essential Bibliografy

Balacheff N., *Conception, connaissance et concept*, Didactique et tecnologies cognitives en mathématiques, séminaires, Université Joseph Fourier (Grenoble 1), 1994-1995, pp. 219-244.

Brousseau G., *Fondaments et méthodes de la didactique des mathématiques*, Recherchesen didactique des mathématiques, Paris, 1986.

D'Amore B., Elementi di Didattica della Matematica, Pitagora Editrice, 1999.

Gras R., *Les fondaments de l'analyse statistique implicative*, Quaderni di Ricerca in Didattica, n. 9, Palermo, 1999.

Knuth E. (2002) Teachers' Conceptions of Proof in the Context of Secondary School Mathematics *Journal of Mathematics Teacher Education* 5 (1) pp. 61-88

Pedemonte B., *Some cognitive aspects of the relationship between argumentation and proof in mathematics*, Proceedings of the 25th conference of the international group for the Psychology of Mathematics Education Utrecht, Editor: van den Heuvel-Panhuizen M., 1999.

Pedemonte B. (2002) *Etude didactique et cognitive des rapports de l'argumentation et de la démonstration dans l'apprentissage des mathématiques,* Thèse de l'Université Joseph Fourier, Grenoble I.

Reiss K., Renkl A.(2002) Learning to Prove: The Idea of Heuristic Examples *Zentralblatt für Didaktik der Mathematik* 34 (1) pp. 29-35

Spagnolo F., Insegnare le matematiche nella scuola secondaria, La Nuova Italia Editrice, Ith Ed. 1998, 1999 (the first reprint).

Spagnolo F., *The role of history of Mathematics in Research in Mathematics Education*, Amman (Giordania), 18-23 Novembre 2000, International Conference on Mathematics Education into the 21st Century.